Log-correlated fields: Some useful tools

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Plan

① Log-correlated field? Examples

② Typical questions

③ Methods:

- (Path) localization, first + second moment method
- Genealogical structure: Seperation of scales
- Gaussian comparison / Berry-Essen bounds
- Non Gaussian?

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① Log-correlated fields

Abstractly speaking: V_n metric space with distance d

$(X_v(n), v \in V_n)$ log-correlated

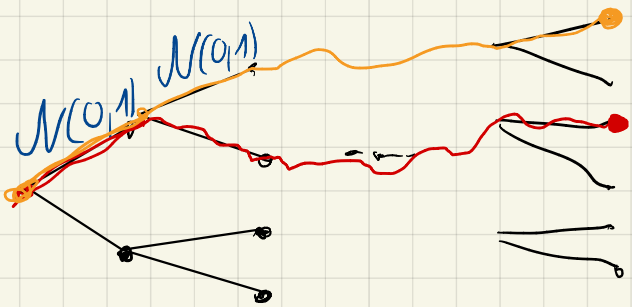
iff $\mathbb{E}(X_v(n) X_{v'}(n)) \approx -\log(d(v,v'))$

for $v, v' \in V_n$.

$\rightsquigarrow$ Slow decay of correlations

Examples

Ex 1 Binary branching random walk with Gaussian increments.



2^n leafs.

on each edge iid $\mathcal{N}(0,1)$.

- V_n = set of leafs.

$X_v(n)$ = sum of weights on path to root $v \in V_n$

$X_v(n) \sim \mathcal{N}(0,n)$

Correlation:

$$\mathbb{E}(X_v(n)\, X_{v'}(n))$$

$=$ # edges that {path $v \to$ root} and {path $v' \to$ root} share.

$=: v \wedge v'$ overlap of v and v', branching time.

Some nice properties:

- Decomposition in indep. increments
- branching time: very explicit

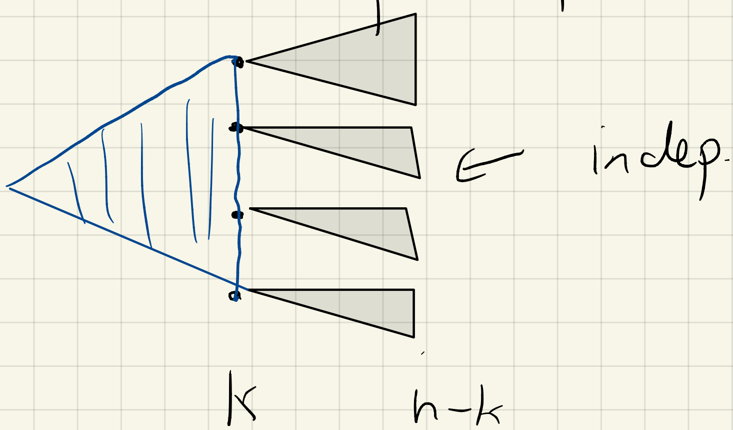
before : full dependence

after : complete independence.

- Self similarity:

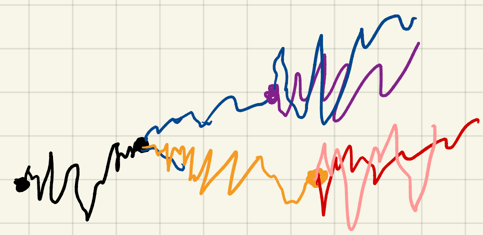
Look at level k:

2^k copies of BRW's with $n-k$ levels



Ex 2 Branching Brownian motion

1. Start on BM at 0.
2. After exp(1) split in two.
3. New particles perform indep BM from splitting location
4. Same splitting rule.



At time t: $n(t)$ number of particles

$\mathbb{E}(n(t)) = e^t$

$\{X_k(t), k \leq n(t)\}$ particle positions.

$X_k(t) \sim \mathcal{N}(0,t)$.

Ex 3 2d DGFF

V_n finite square of $\mathbb{Z}^2$ n indicating the size

$\mathbb{P}_v$: law of SRW S starting from $v \in V_n$

$\mathbb{E}_v$ corresp. expect.

τ_n exit time of V_n.

$\{X_v(n), v \in V_n\}$ Gaussian process with mean zero and covariance

$$\mathbb{E}[X_v(n) X_{v'}(n)] = \mathbb{E}_v\Big[\sum_{k=0}^{\tau_n} \mathbb{1}_{S_k = v'}\Big]$$

$$= G_n(v, v')$$

As $d=2$: G_n behaves like log!

Ex 4 Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ primes}} (1-p^{-s})^{-1}$$

critical axis : $\frac{1}{2} + it$, $t \in \mathbb{R}$.

Choose an interval of length 1 uniformly from $[0, T]$ large.

Look at $\{\zeta(\frac{1}{2}+it), t \in \text{random int.}\}$.

This looks very much like a log-correlated fields!

Ex 5 Log of characteristic polynomial for (some) random matrix ensembles

For example: CUE
$N \times N$ matrix unitary sampled uniformly from the unitary group

$P_{U_N}(\theta)$ characteristic polynomial

$\log |P_{U_N}(\theta)|$ is essentially log-correlated

② What are questions of interest?

① Level sets

② Extreme values

- Rough order of maximum.
- max $\approx$ Expression + $O(1)$ fluctuations
- Extremal process
- Quantitative results

③ Gaussian multiplicative chaos.

$$"e^{\gamma \cdot \mathrm{Field}(x)} dx"$$

Size parameter $\to \infty$: Convergence when properly normalized?

If $\gamma < \gamma^c$: Limit object GMC

③ Extremes

- $\{X_v(n)\}$ binary BRW n-levels.
- Upper bound on $\max_{v \in V_n} X_v(n)$:

$$\mathbb{P}\left(\max_{v \in V_n} X_v(n) > y\right)$$

$$= \mathbb{P}\left(\sum_{v \in V_n} \mathbb{1}_{\{X_v(n) > y\}} \geq 1\right)$$

$$\overset{\text{Markov}}{\leq} \mathbb{E}\left(\sum_{v \in V_n} \mathbb{1}_{\{X_v(n) > y\}}\right)$$

$$= \sum_{v \in V_n} \mathbb{P}(\underbrace{X_v}_{\sim \mathcal{N}(0,n)}(n) > y)$$

$$= 2^n \, \mathbb{P}(\underbrace{X}_{\sim \mathcal{N}(0,n)}(n) > y)$$

$$\overset{\text{Gaussian tail}}{\sim} 2^n \frac{\sqrt{n}}{\sqrt{2\pi}\, y} e^{-y^2/2n} = \frac{\sqrt{n}}{\sqrt{2\pi}\, y} e^{\log 2 \, n - y^2/2n}$$

Find y such that this is of order 1.

$$y = \sqrt{2 \log 2}\, n \quad + o(n)$$

Good choice: $y = \sqrt{2\log 2}\, n - \frac{1}{2\sqrt{2 \log 2}} \log n$

If C is large constant and we take

$$y = \sqrt{2\log 2}\, n - \frac{1}{2\sqrt{2\log 2}} \log n + C$$

Then

$$\mathbb{P}\left(\max_{v \in V_n} X_v(n) > y\right) \quad \text{small!}$$

Result:

$$\sqrt{2\log 2}\, n - \frac{1}{2\sqrt{2\log 2}} \log n + O(1)$$

first upper bound on max.

<u>Note</u> We have not used anything about BRW!

Apart from: 2^n r.v.

Each r.v. $\mathcal{N}(0,n)$

For example: Same computation with 2^n iid $\mathcal{N}(0,n)$.

In fact: $\sqrt{2\log 2}\, n - \frac{1}{2\sqrt{2\log 2}} \log n + O(1)$

is the order of max. of 2^n iid $\mathcal{N}(0,n)$.

<u>In fact</u> True answer for BRW:

$$\max \approx \sqrt{2\log 2}\, n - \frac{3}{2\sqrt{2\log 2}} \log n + O(1)$$

What we have not used:

k level $\rightsquigarrow$ 2^k Gaussians with variance k.

$\hookrightarrow$ Try to do better tomorrow!

<u>Universality:</u> Log correlated models

$\leftrightarrow$ iid set-up

max. have same first order and

in the second order replace $1 \leftrightarrow 3$!