Comparison

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Idea
- BRW / BBM direct analysis of the extremes
- Gaussian processes: we have more tools

Tools to compare different Gaussian processes | Comparing Gaussian models with non-Gaussian

Gaussian comparison

- Two indep. Gaussian processes, mean zero,

  "one is more correlated than other"

  $\longrightarrow$ max of one is larger then max of other

Example Binary BRW 2^n iid Gaussians

Cov: overlap.   two diff: 0

Max $\leq$

Kahanes Thm: X, Y n-dim Gaussian X, Y indep. mean zero.

$D_1, D_2 \subseteq \{1, \dots, n\} \times \{1, \dots, n\}$

$$\mathbb{E}(X_i X_j) \geq \mathbb{E}(Y_i Y_j) \qquad (i,j) \in D_1$$

$$\mathbb{E}(X_i X_j) \leq \mathbb{E}(Y_i Y_j) \qquad (i,j) \in D_2$$

$$\mathbb{E}(X_i X_j) = \mathbb{E}(Y_i Y_j) \qquad (i,j) \notin D_1 \cup D_2$$

F: on $\mathbb{R}^n$, moderate growth, twice diff.
and

$$\frac{\partial^2}{\partial x_i \partial x_j} F(x) \geq 0 \qquad (i,j) \in D_1$$

$$\frac{\partial^2}{\partial x_i \partial x_j} F(x) \leq 0 \qquad (i,j) \in D_2$$

Then

$$\mathbb{E}(F(X)) \leq \mathbb{E}(F(Y))$$

Rem • One very useful function:

Laplace transforms $e^{\lambda x}$, $\lambda > 0$

• $F = \max\limits_{i=1,\dots,n} X_i$ (?)

Slepian's Lemma

$$\mathbb{E}(X_i X_j) \geq \mathbb{E}(Y_i Y_j) \qquad \forall i \neq j$$
$$\mathbb{E}(X_i X_i) = \mathbb{E}(Y_i Y_i) \qquad \forall i$$

$$\Rightarrow \mathbb{E} \max X_i \leq \mathbb{E} \max Y_i.$$

Rem • In particular:

Gaussian process 1 − Cov that you understand $\leq$ Model cov $\leq$ Gaussian process 2 Cov that you understand

- There are more functions connected to maximum, where such a result holds.

Methods used to prove them (useful on its own)

(Kahane Thm)

Build X^h whose covariance interpolates between the one of X and Y.

$$X^h := \sqrt{h}\, X + \sqrt{1-h}\, Y$$

$$\mathrm{Cov}(X^h) = h\,\mathrm{Cov}(X) + (1-h)\,\mathrm{Cov}(Y)$$

- $f(h) = \mathbb{E}\, F(X_1^h, \dots, X_n^h)$

$$f(1) - f(0) = \int_0^1 dh\, \underbrace{\frac{d}{dh} f(h)}_{\text{Explicit.}}.$$

- Gaussian integration by parts:

g moderate growth $\mathbb{R}^n \to \mathbb{R}$

$$\mathbb{E}\, g(X) X_i = \sum_{j=1}^n \underbrace{\mathbb{E}(X_i X_j)}_{\text{magic where the covariance appears!}} \overbrace{\mathbb{E}\left[\frac{\partial}{\partial x_j} g(X)\right]}^{\leadsto \text{ assume to know sign}}.$$

Work with functions that are not twice differentiable:

Approximate them.

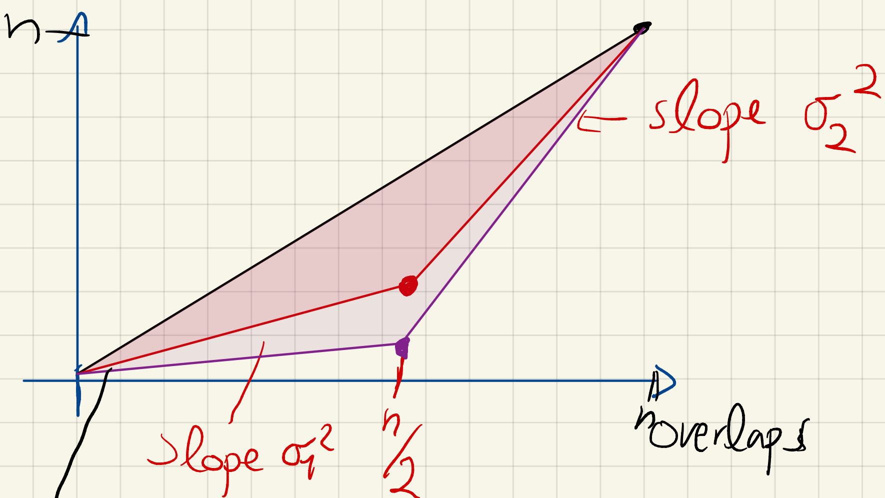
One application

Two-speed BRW:

First $\frac{n}{2}$ levels $\mathcal{N}(0, \sigma_1^2)$

The other $\frac{n}{2}$ level $\mathcal{N}(0, \sigma_2^2)$

s.t. $\frac{n}{2}\sigma_1^2 + \frac{n}{2}\sigma_2^2 = n$



Representation of Covariance as a function of overlap

Gaussian comparison:

$\mathbb{E}$ of maxima are ordered!

"Lower functions have higher max".

Comparing with Gaussian model with non-Gaussian

1) Localization works for non-Gaussian things

Berry-Essen bound

Thm Let $(W_i, i \geq 1)$ sequence of indep. random vectors on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), P)$ with mean $E(W_i)$ and cov matrix $Cov(W_i)$.

Define

$$\mu_m = \sum_{i=1}^m \mathbb{E} W_i, \quad \Sigma_m = \sum_{i=1}^m Cov\, W_i$$

λ_m smallest eigenvalue of Σ_m

Q_m law of $W_1 + \dots + W_m$.

Then there exists a constant c (only dep d) s.t

Gaussian measure with mean μ_m, cov Σ_m

$$\sup_{A \in \mathcal{A}} \left| Q_m(A) - \eta_{\mu_m, \Sigma_m}(A) \right|$$

collection of convex Borel meas. subsets of $\mathbb{R}^d$

$$\leq c \lambda_m^{-3/2} \sum_{i=1}^{m} \mathbb{E}[\|W_i - \mathbb{E}[W_1]\|^3]$$

Idea why it could be useful

If you are able to write your model in terms of sum of indep. increments.
not identically distr.

[increments $\to W_i$]

Then: compare to corresponding Gaussian measure!

$\hookrightarrow$ e.g. Gaussian branching random walk.

Difficulty: Need good bounds for r.h.s!

This method has been succesfully used
Arguin, Belius, Harper "Maxima of a randomized Riemann zeta function".

$\longrightarrow$ Toolbox to compare different models.

Useful Tools

→ localization

→ first + second moments

→ Comparing Gaussian models via Gaussian compare

→ to non-Gaussian ones!

[Still is a lot to be understood ...]