Schramm-Loewner evolutions and imaginary geometry

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Outline

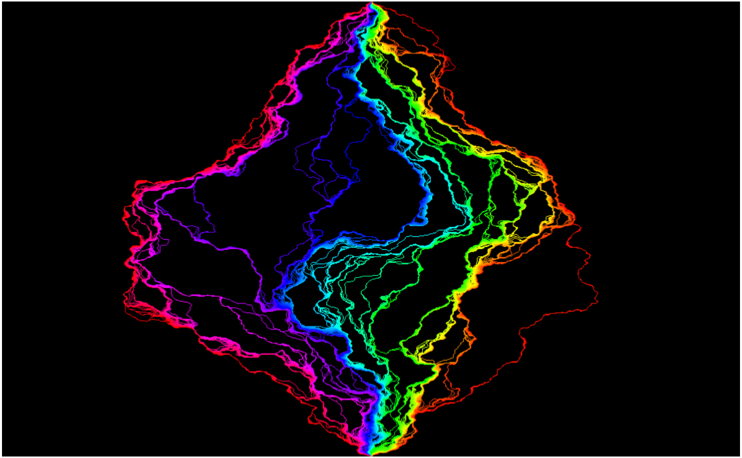
- Lecture 1: Definition and basic properties of SLE, examples
- Lecture 2: Basic properties of SLE
- **Lecture 3: Imaginary geometry (today)**

References:
Conformally invariant processes in the plane by Lawler
Lectures on Schramm-Loewner evolution by Berestycki and Norris
Imaginary geometry I: Interacting SLEs by Miller and Sheffield

Note: Many of today's figures are from Miller and Sheffield's papers

Imaginary geometry

- Framework for constructing natural couplings of multiple SLEs



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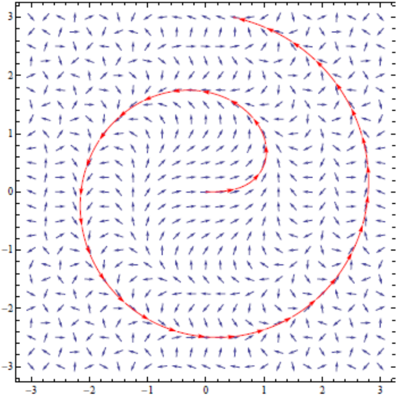

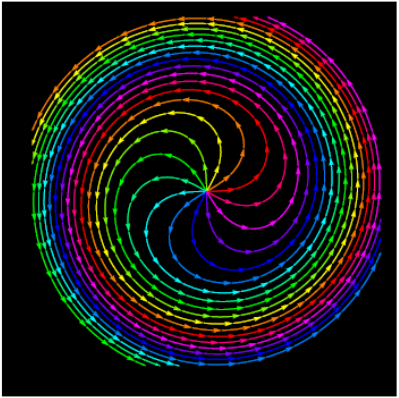

η satisfying $\eta'(t) = e^{ih(\eta(t))}$, $h(z) = |z|^2$

Flow lines of $e^{i(h(\eta(t))+\theta)}$

Imaginary geometry

- Framework for constructing natural couplings of multiple SLEs
- An SLE_κ for $\kappa \in (0,4)$ is a flow line η satisfying

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where $\chi = 2/\sqrt{\kappa} - \sqrt{\kappa}/2$ and h is the **Gaussian free field**.

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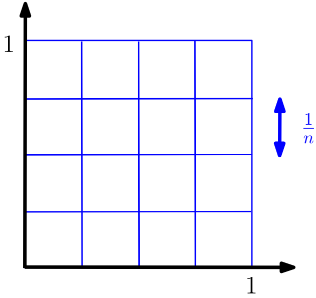
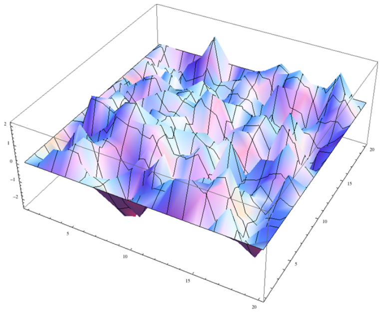
- This definition is only a heuristic since h is a **generalized function (distribution)** rather than a true function.
- Theory developed by Dubédat and Miller-Sheffield.

The discrete Gaussian free field

- Hamiltonian $H(f)$ quantifies deviation of f from being harmonic

$$H(f) = \frac{1}{2}\sum_{x\sim y}(f(x) - f(y))^2, \qquad f : \frac{1}{n}\mathbb{Z}^2 \cap [0,1]^2 \to \mathbb{R}.$$





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  - $h_n|_{\partial[0,1]^2} = g$ for given boundary data g,
  - prob. density rel. to prod. of Lebesgue measure prop. to

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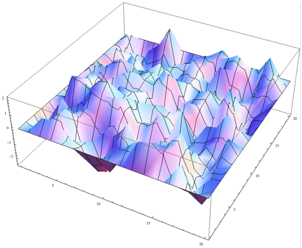

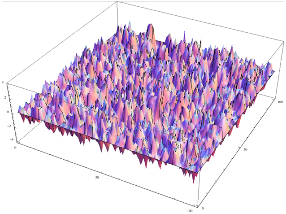

$n = 20, \; n = 100$

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$$\exp(-H(h_n)).$$

- If g also denotes the discrete harmonic extension of the boundary data and $z, w \in (0,1)^2$ are fixed,

$$h_n(z) \sim \mathcal{N}(g(z), \log n + O(1)),$$
$$\mathrm{Cov}(h_n(z), h_n(w)) = \log|z-w|^{-1} + O(1).$$

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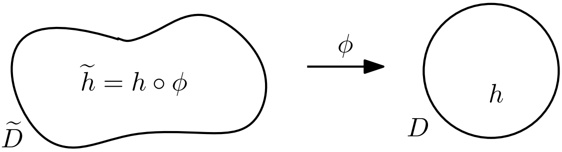

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- Domain Markov property: For $U \subset D$ open, conditioned on $h|_{D \setminus U}$ the law of $h|_U$ is that of $h_0 + \mathfrak{h}$, where
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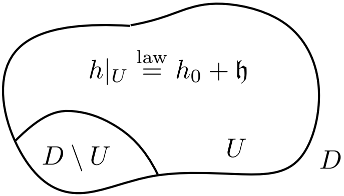

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  - h_0 is a zero-boundary GFF in U and
  - $\mathfrak{h}$ is the harmonic extension of $h|_{\partial U}$ to U.
- The GFF is uniquely characterized by conformal invariance and domain Markov property, plus a moment assumption (Berestycki-Powell-Ray'20).

Flow lines of the Gaussian free field

Goal: solve $\eta'(t) = e^{ih(\eta(t))/\chi}$, $\chi > 0$.

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Natural approach which we will **not** take:

- Let h_ϵ be a regularized version of h.
- Solve $\eta'(t) = e^{ih_\epsilon(\eta(t))/\chi}$.
- Send $\epsilon \to 0$ and argue that η converges.

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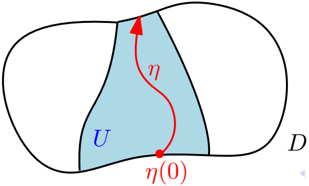
Instead we ask: Inspired by the case when h is smooth, which properties is it natural to require that η satisfies?

Examples:

- Locality: To determine whether $\eta \subset U$ it is sufficient to observe $h|_U$.
- Coordinate changes (next slide).



Coordinate change

Suppose h is smooth and η solves

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Then $\widetilde{\eta}(t) := \phi^{-1}(\eta(t))$ solves

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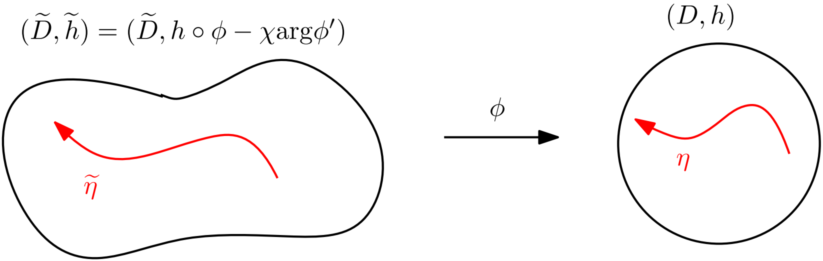

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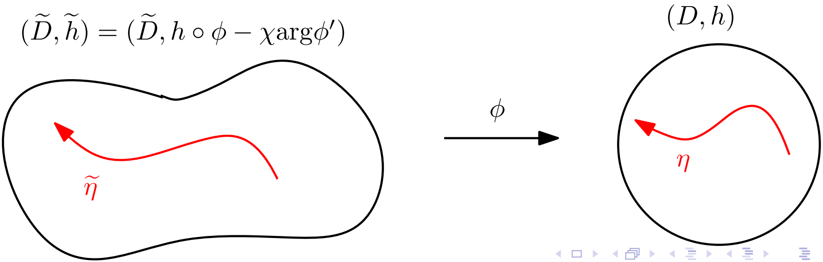
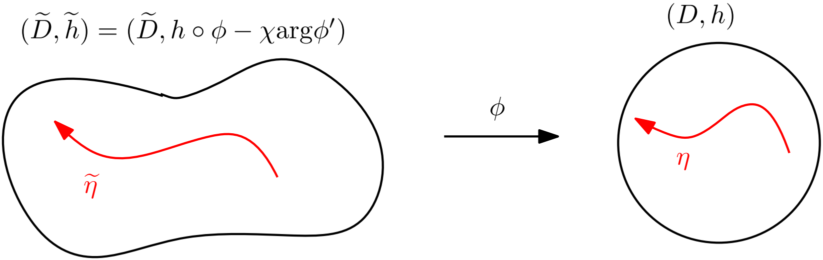
Proof by chain rule with $\psi = \phi^{-1}$:

$$\widetilde{\eta}'(t) = \frac{d}{dt}(\psi \circ \eta(t)) = \psi'(\eta(t))\eta'(t) = \psi'(\eta(t))e^{ih(\eta(t))/\chi} = e^{i\widetilde{h}(\eta(t))/\chi}.$$



Coordinate change



We say that (D, h) and $(\widetilde{D}, \widetilde{h})$ are **equivalent**.

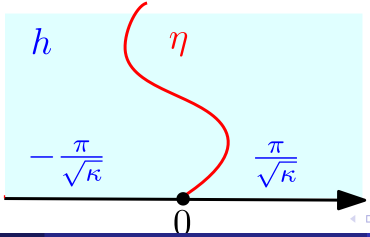
$(D, h) \overset{\phi}{\equiv} (\widetilde{D}, \widetilde{h})$.

Note! The equivalence relation also makes sense for h not smooth.

SLE as a flow line of the GFF

Theorem (Dubédat'09, Miller-Sheffield'16)

1. *For $\kappa > 0$, the GFF h determines a curve η with the law of an SLE_κ on $(\mathbb{H}, 0, \infty)$ such that the following hold.*
2. *Locality: The event $\eta \cap U = \emptyset$ determined by $h|_{\mathbb{H}\setminus U}$ for $U \subset \mathbb{H}$ open.*
3. *Coordinate change and domain Markov property: For any stopping time τ for η define h^τ such that the following holds*
$$(\mathbb{H} \setminus K_\tau, h|_{\mathbb{H}\setminus K_\tau}) \overset{g_\tau}{\equiv} (\mathbb{H}, h^\tau).$$
Then the conditional law of h^τ given $\eta|_{[0,\tau]}$ is equal to the law of h.



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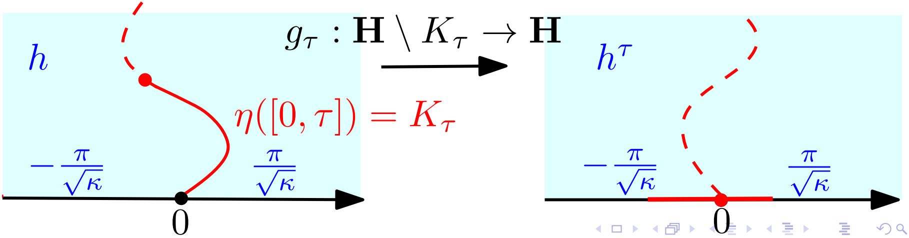

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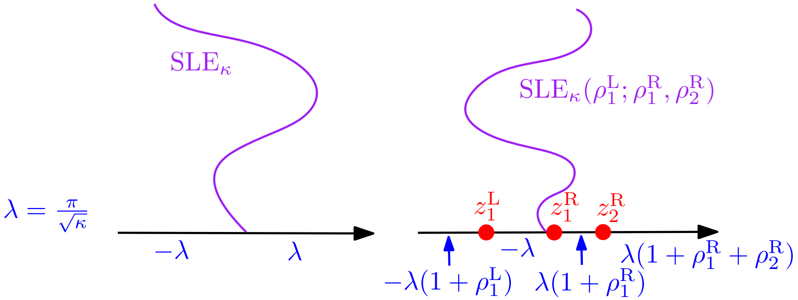
Proof idea:

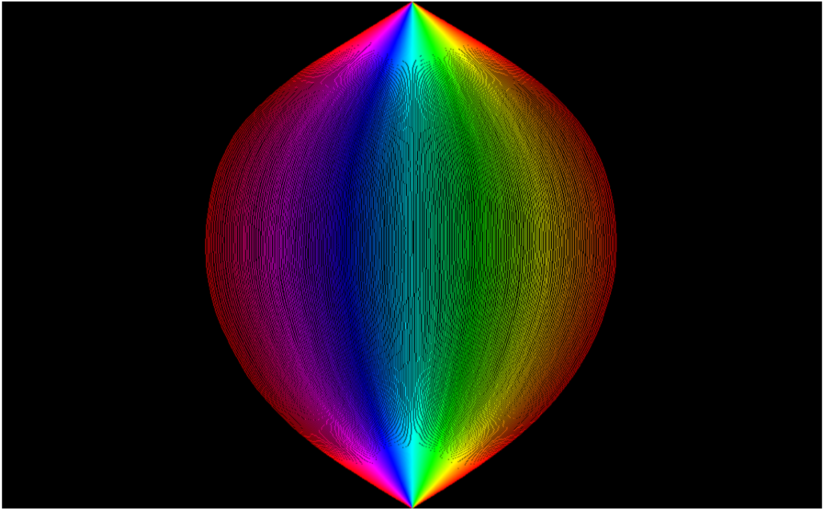
1. Construct a coupling (h, η) satisfying variants of 2. and 3.
2. Prove that in this coupling h determines η.

Flow lines with piecewise constant boundary data: $\mathrm{SLE}_\kappa(\underline{\rho})$



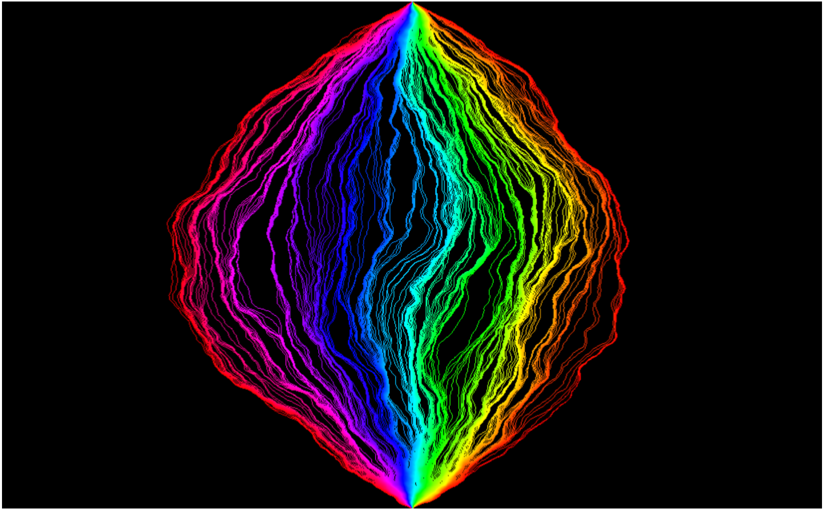
- $\mathrm{SLE}_\kappa(\underline{\rho})$ with $\underline{\rho} = (\rho_1^{\mathrm{L}}, \ldots, \rho_{n_{\mathrm{L}}}^{\mathrm{L}}; \rho_1^{\mathrm{R}}, \ldots, \rho_{n_{\mathrm{R}}}^{\mathrm{R}})$ are variants of SLE_κ with force points at $(z_1^{\mathrm{L}}, \ldots, z_{n_{\mathrm{L}}}^{\mathrm{L}}; z_1^{\mathrm{R}}, \ldots, z_{n_{\mathrm{R}}}^{\mathrm{R}})$ which are either repulsive $(\rho_j^{\mathrm{L}}, \rho_j^{\mathrm{R}} > 0)$ or attractive $(\rho_j^{\mathrm{L}}, \rho_j^{\mathrm{R}} < 0)$.
- Can also be defined easily with Loewner chains by modifying the driving function.

Simulations



Flow lines of $e^{i(h/\chi+\theta)}$ for $\theta \in [-\pi/2, \pi/2]$, $\kappa = 1/4096$

Simulations



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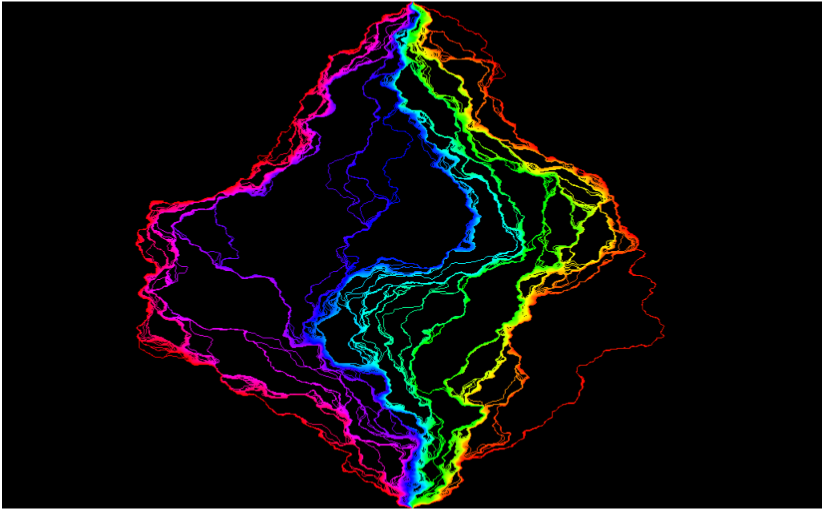

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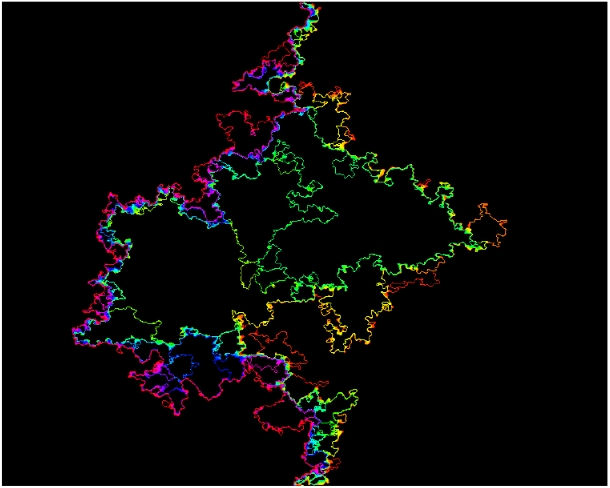
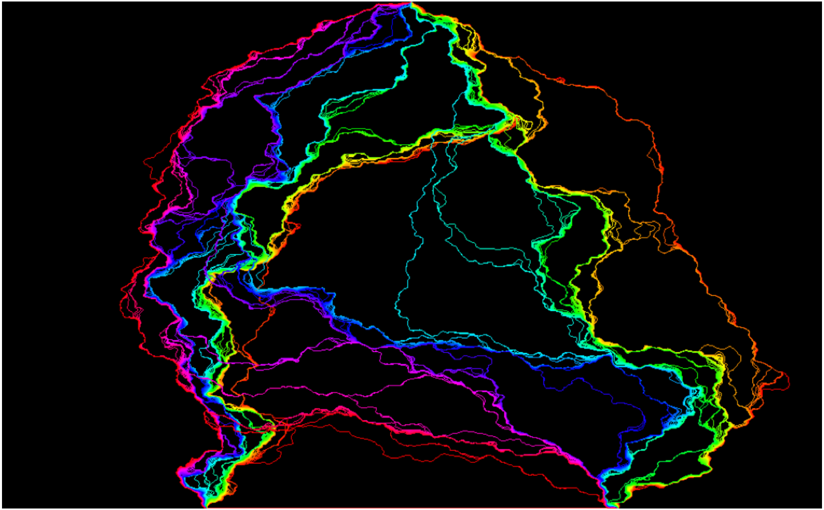


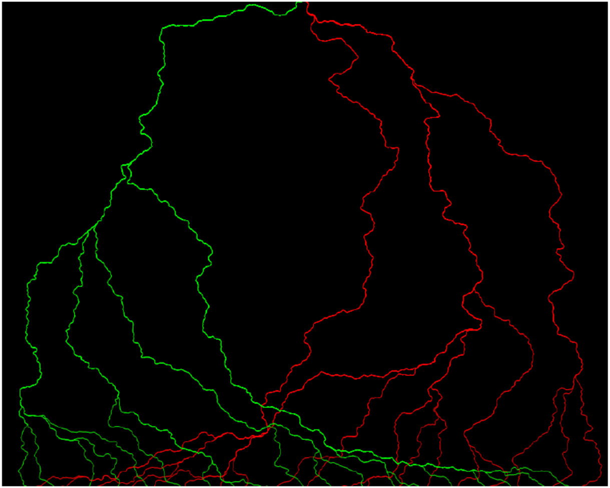
Flow lines of $e^{i(h/\chi+\theta)}$ for $\theta \in [-\pi/2, \pi/2]$, $\kappa = 2$

Simulations



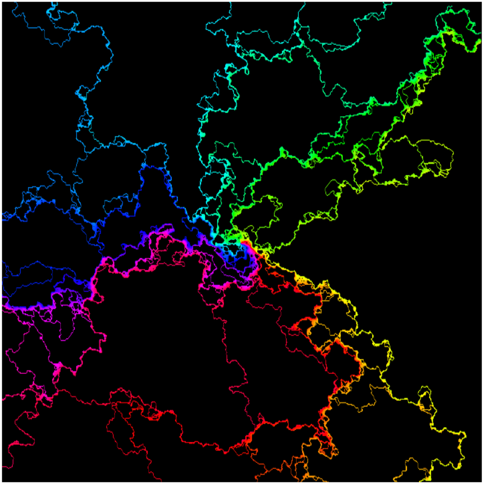
Flow lines of $e^{i(h/\chi+\theta)}$ for $\theta \in [-\pi/2, \pi/2]$, $\kappa = 1/2$, two starting points

Simulations



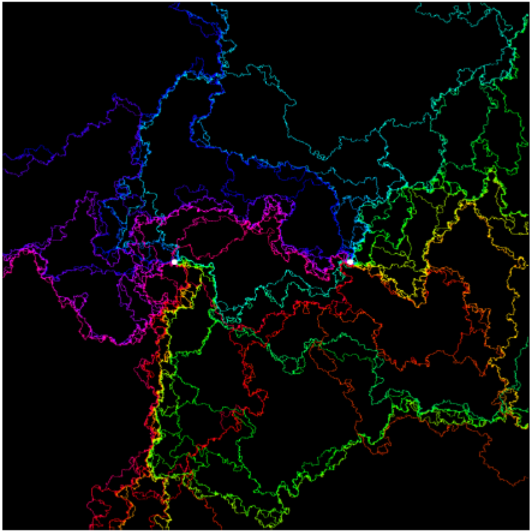
Flow lines of $e^{i(h/\chi+\theta)}$ for $\theta = \pi/4$ (green) and $\theta = -\pi/4$ (red), $\kappa = 1/2$

Simulations



Flow lines of $e^{i(h/\chi+\theta)}$ for $\theta \in [0, 2\pi)$, $\kappa = 4/3$, started from interior point

Simulations



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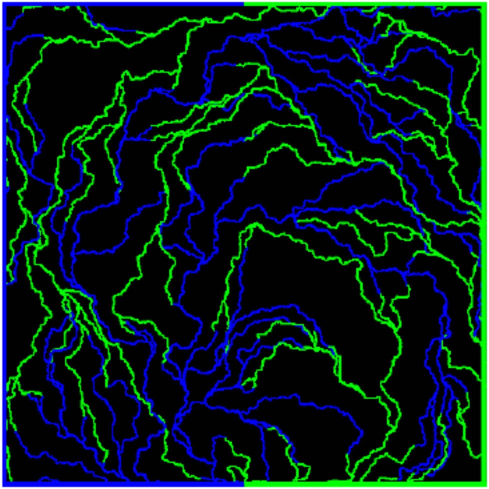
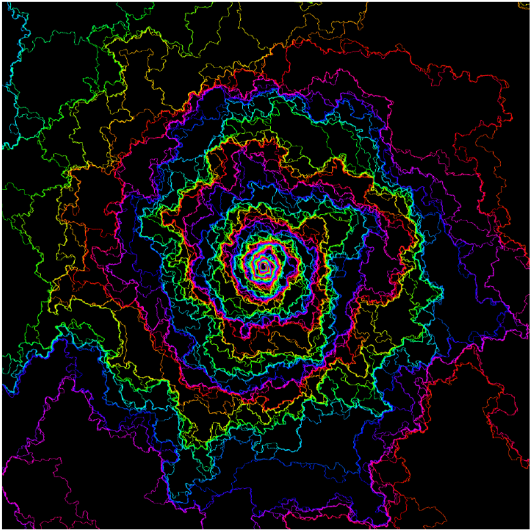


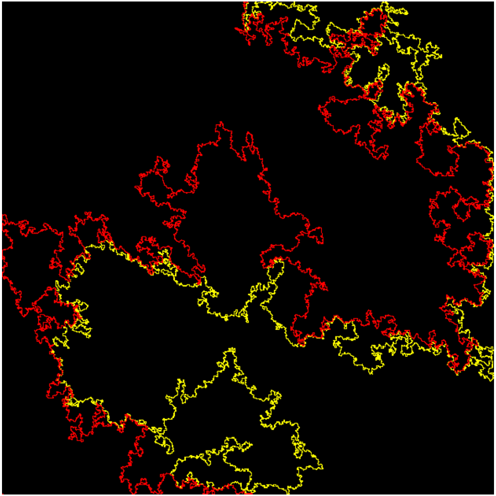
Flow lines of $e^{i(h/\chi+\theta)}$ for $\theta = \pi/2$ (blue) and $\theta = -\pi/2$ (green), $\kappa = 1/2$, started from 100 uniformly chosen points

Simulations



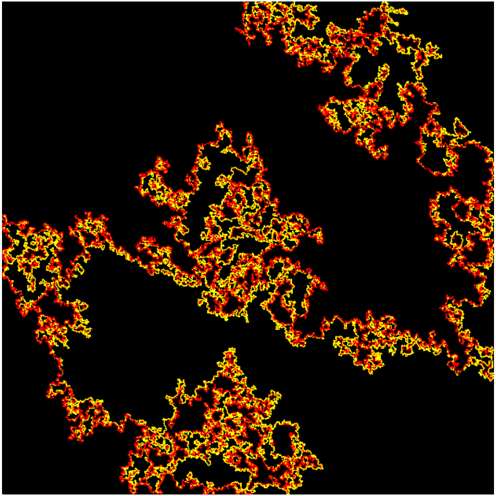
Flow lines of $e^{i(h/\chi+\theta)}$ for $\theta \in [0, 2\pi)$ and $h =$GFF$-5\log|z|$.

Simulations



Flow lines of $e^{i(h/\chi+\theta)}$ for $\theta = \pm\pi/2$, $\kappa = 3$

Simulations



Flow lines of $e^{i(h/\chi+\theta)}$ for θ varying and piecewise constant $\pm\pi/2$, $\kappa = 3$. The union of these flow lines have the law of $\mathrm{SLE}_{16/3}$!

Recall: SLE as a flow line of the GFF

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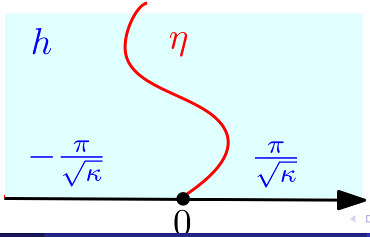

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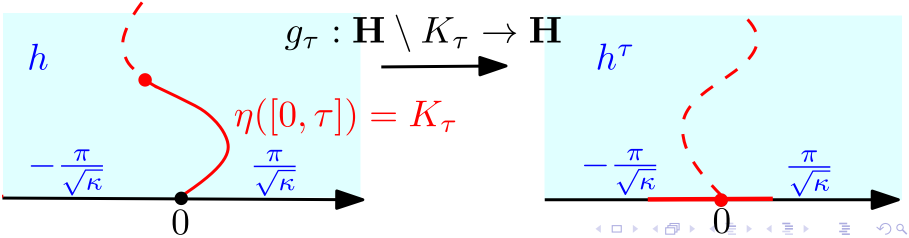

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Proof idea:

1. Construct a coupling (h, η) satisfying variants of 2. and 3.
2. Prove that in this coupling h determines η.

Constructing a coupling of GFF and SLE

Proposition

1. *There is a coupling of a GFF h and an SLE_κ η s.t. the following hold.*
2. *Locality: $\mathbb{P}[\eta \cap U = \emptyset \mid h]$ is a function of $h|_{\mathbb{H}\setminus U}$ for $U \subset \mathbb{H}$ open.*
3. *Coordinate change and domain Markov property: For any stopping time τ for η define h^τ such that the following holds*
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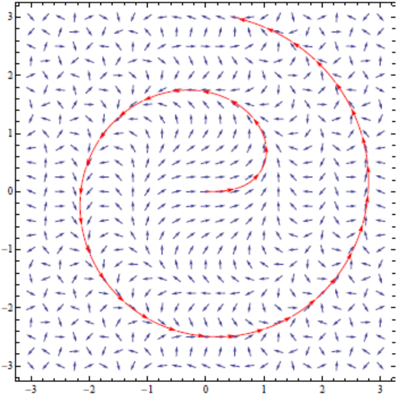
- Let η be an SLE_κ; let $\mathfrak{h}_t$ denote the harmonic extension of h from $\partial(\mathbb{H} \setminus K_t)$ to $\mathbb{H} \setminus K_t$ if proposition holds.
- Loewner equation and Itô calculus give that for each fixed $z \in \mathbb{H}$, the process $\mathfrak{h}_t(z)$ is a local cts martingale with explicit quadratic variation.
- Same statement for $(\mathfrak{h}_t, \phi)$ instead of $\mathfrak{h}_t(z)$ until $K_t \cap \mathrm{supp}(\phi) \neq \emptyset$.
- $(\mathfrak{h}_\tau + \widetilde{h})|_V \stackrel{d}{=} h|_V$ if $K_\tau \cap V = \emptyset$ a.s., where $\widetilde{h}$ is 0-bdy GFF in $\mathbb{H} \setminus K_\tau$.
- Extend coupling to multiple stopping times τ of η and domains V.

GFF values along the SLE

What are the values of h along η?

Recall: $\eta'(t) = e^{ih(\eta(t))/\chi}$.



$h(z) = |z|^2$. On η: $h = \chi \cdot \text{winding} \pmod{2\pi\chi}$.

GFF values along the SLE

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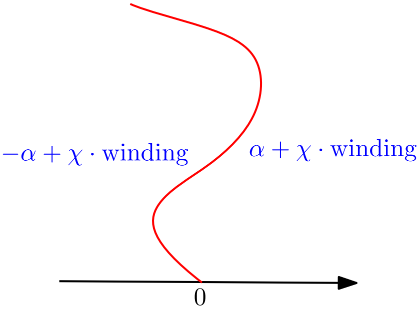

h GFF, η SLE. Angle gap $= 2\alpha$, $\alpha = \frac{\pi\sqrt{\kappa}}{4}$

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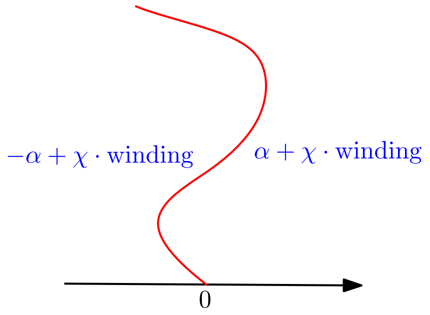

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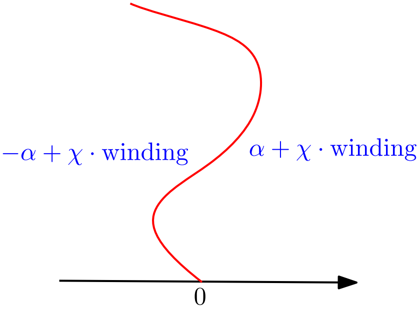

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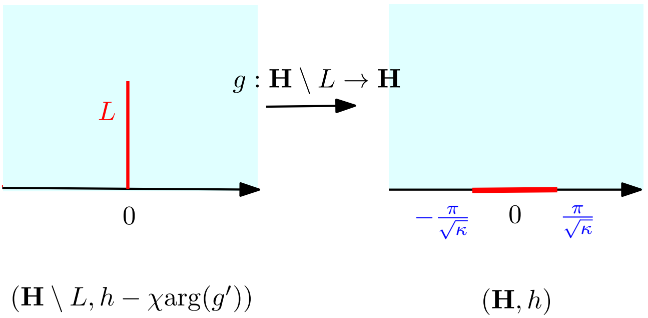
SLE has infinite winding, but harmonic extension of bdy data well-defined.

Case $\kappa = 4$, $\chi = 0$: SLE_4 level lines of GFF (Schramm-Sheffield'09)

GFF values along the SLE

Domain Markov property (with L instead of η): Conditioned on L, $h^\tau \stackrel{d}{=} h$.



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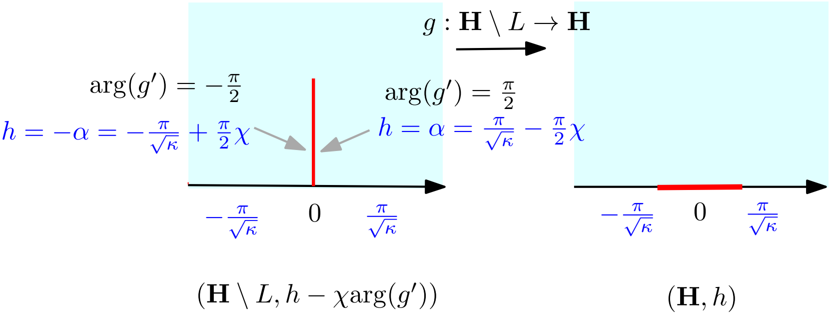

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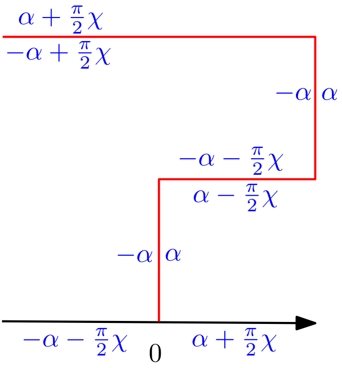
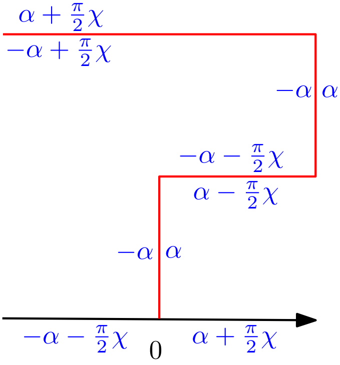


Left: $-\alpha + \chi \cdot \text{winding}$. Right: $\alpha + \chi \cdot \text{winding}$. Angle gap $= 2\alpha = \frac{\pi\sqrt{\kappa}}{2}$.

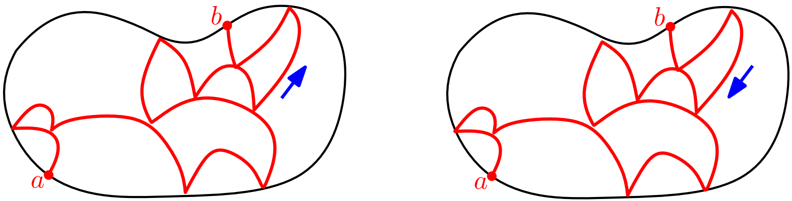
GFF values along the SLE



Left: $-\alpha + \chi \cdot$ winding. Right: $\alpha + \chi \cdot$ winding. Angle gap $= 2\alpha = \frac{\pi\sqrt{\kappa}}{2}$.

A few examples of applications

- reversibility of SLE_κ for $\kappa \in (4, 8)$ (Miller-Sheffield'16)



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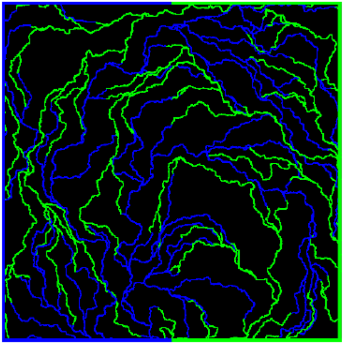

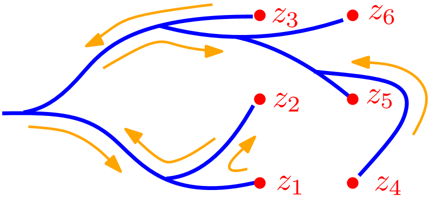

The space-filling $\text{SLE}_{16/\kappa}$ visits the points in this order: $z_1, z_2, z_5, z_4, z_6, z_3$

Left: Flow lines of $e^{i(h/\chi+\theta)}$ for $\theta = \pi/2$ (blue) and $\theta = -\pi/2$ (green), $\kappa = 1/2$, started from 100 uniformly chosen points

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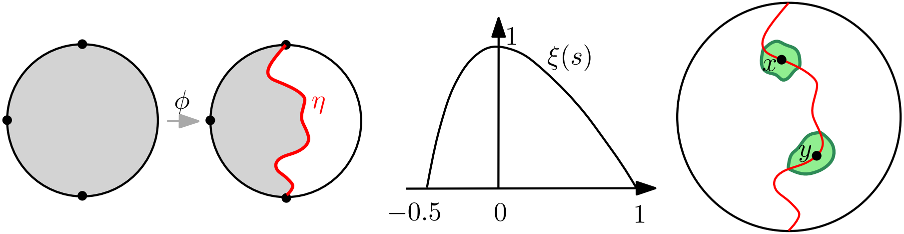

$$\xi(s) = \dim_{\mathcal{H}} \left(\{x \in \partial\mathbb{D} \, : \, \phi'((1-\epsilon)x) = \epsilon^{-s+o(1)}\} \right)$$

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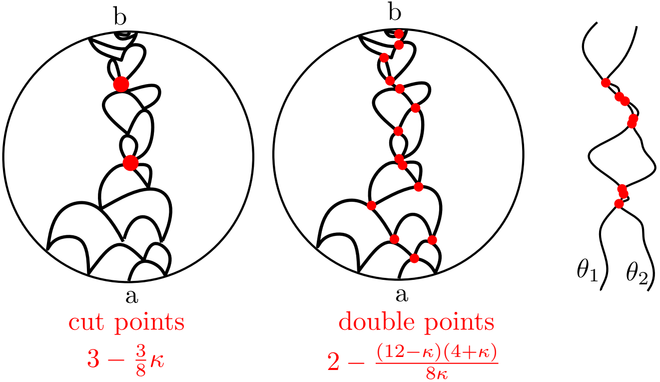

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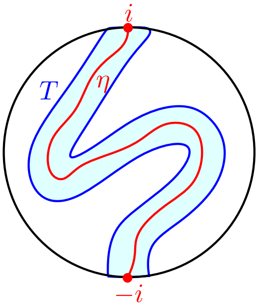

Lemma

$\mathbb{P}[\eta \subset T] > 0$ *for* η *SLE*$_\kappa$ *on* $(\mathbb{D}, -i, i)$

- η flow line of GFF h in $\mathbb{D}$.
- $\widetilde{\eta}$ SLE_κ on $(T, -i, i)$ and flow line of GFF $\widetilde{h}$ in T.
- η abs. cts. w.r.t. $\widetilde{\eta}$ since $h|_T$ abs. cts. w.r.t. $\widetilde{h}$ away from ∂T.

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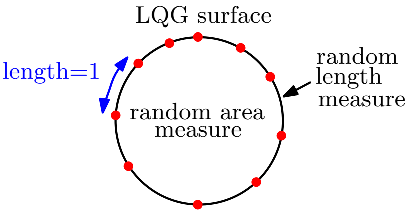

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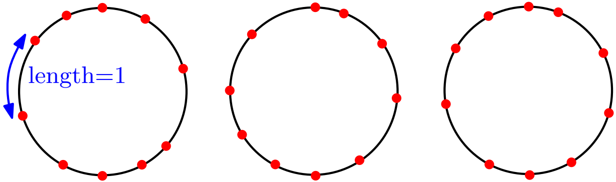

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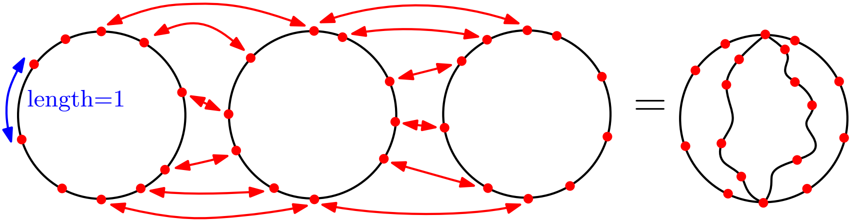

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Thanks for attending!