Uniform Spanning Trees in High Dimension

T. Hutchcroft OOPS 2020.

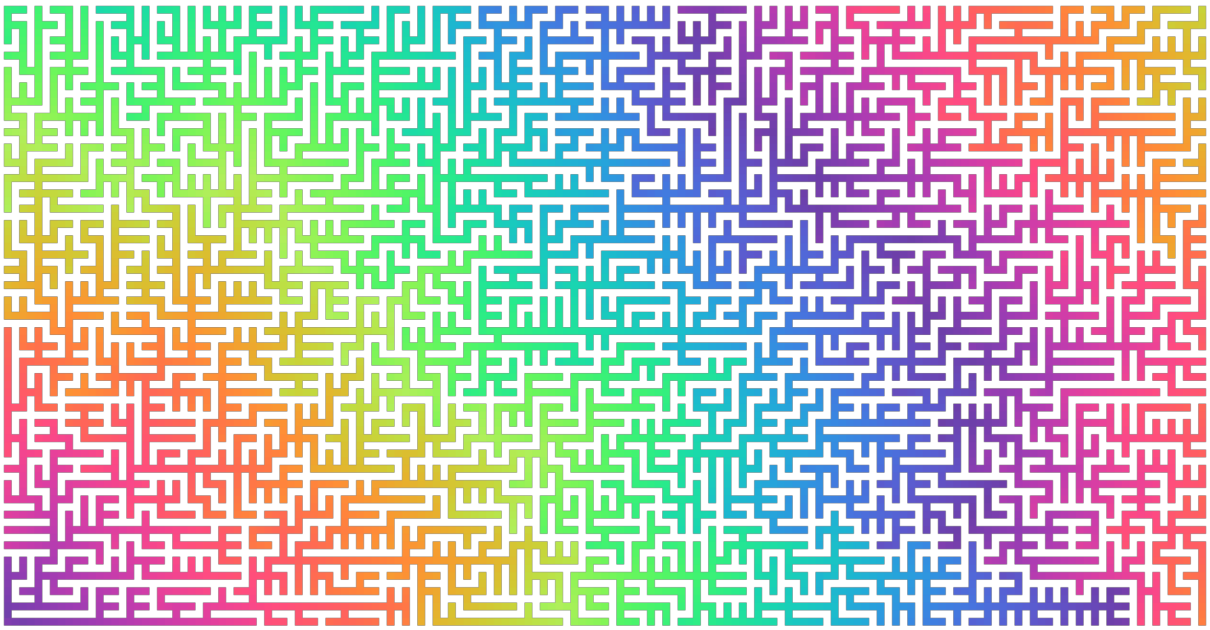


image due to Mike Bostock

Outline of lectures:

1: Basics + Sampling Algorithms

2: The connectivity / disconnectivity transition

3: The interlacement Aldous–Broder Algorithm

4: Critical exponents in high dimensions.

§1: Basics + Sampling Algorithms

$G = (V, E)$ finite connected graph

$\mathcal{T}(G)$ set of spanning trees of G.

connected subgraphs of G containing every vertex and no cycles.

"Uniform spanning tree" — Uniform random element of $\mathcal{T}(G)$.

Turns out to be a very interesting object with connections to many other topics.

A "statistical mechanics" model that is both non-trivial and (relatively) tractable

- Development of SLE — Schramm, Lawler, Schramm and Werner
- 3d scaling limits — Kozma, Angel, Croydon, Hernandez-Torres, Shiraishi

Sampling algorithms

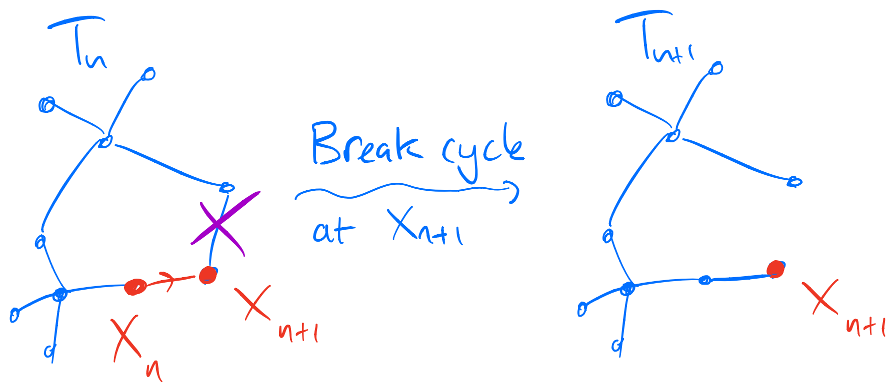
1: MCMC

$G = (V, E)$ finite.

Define a Markov chain (X, T) on $V \times \mathcal{T}$ as follows

- X is a random walk
- Given T_n and X_n, X_{n+1}, define T_{n+1} by adding $\{X_n, X_{n+1}\}$ then "breaking the cycle at X_{n+1}"

"UST Chain"



(If $\{X_n, X_{n+1}\} \in T_n$, $T_{n+1} = T_n$)

Markov-Chain-Tree Theorem:
$\pi \otimes \mathrm{UST}$ is the unique stationary measure for the UST chain.

stationary measure for SRW

2: Aldous-Broder Algorithm

$G=(V,E)$ connected finite, $(X_n)_{n\geq 0}$ random walk on G.

$e(v,X)$ = oriented edge crossed by X as it enters v for first time.

not defined for $v = X_0$.

$$AB(X) = \{ e(v,X)^{\leftarrow} : v \neq o \}$$

reversal

Thm (Broder 1989, Aldous 1990). G finite connected, X random walk started at v. Then $AB(X)$ is distributed as a UST of G oriented towards v.

Proof sketch $(X_n)_{n\in\mathbb{Z}}$ doubly-∞ SRW, $X_0 \sim \pi$.

$\ell_n(v)$ = last edge used to exit v before time n

$e_n(v)$ = first edge used to enter v after time n

Defined for $v \neq X_n$

Observation:

$(X_n, \{\ell_n(v) : v \neq X_n\})$ evolves as UST Chain!

Stationarity + MCTT $\Rightarrow$

by construction.

$$(X_n, \{\ell_n(v) : v \neq X_n\}) \sim \pi \otimes UST$$

$$\Big\updownarrow \quad \text{by reversibility.}$$

$$(X_n, \{e_n(v) : v \neq X_n\})$$

$\square$

Corollary: Paths in the UST are distributed as loop-erased random walks

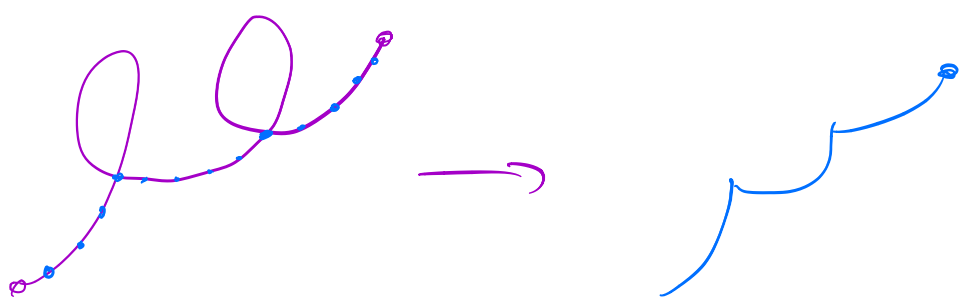
LERW: Let $X = (X_n)_{n=0}^{N}$ either have $N < \infty$ or be transient (visit each vertex at most finitely often).

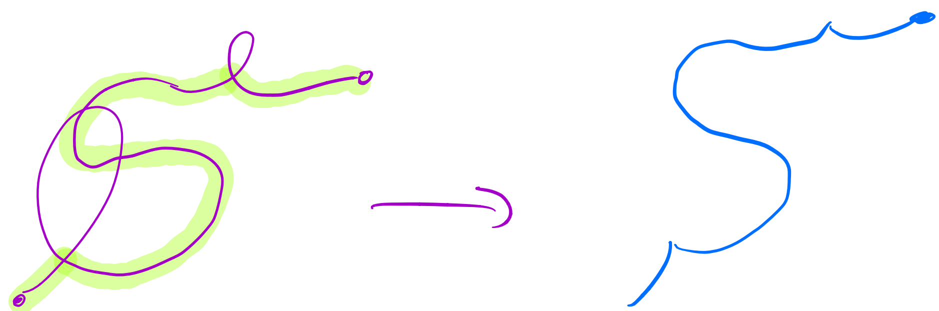
$$\ell_0 = 0 \qquad \ell_{n+1} = 1 + \sup\{ i \geq \ell_n : X_i = X_{\ell_n}\}$$

(Stop when $X_{\ell_n} = X_N$)

$$LE(X) = (X_0, X_{\ell_1}, X_{\ell_2}, \ldots)$$

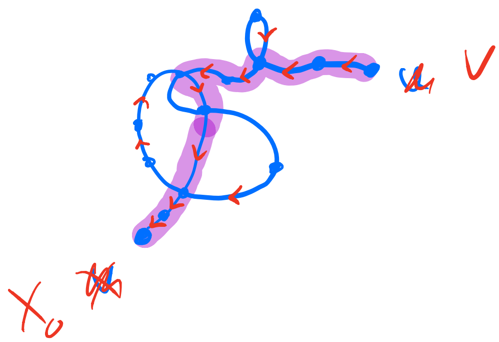
"erase loops chronologically as they are created".





In $AB(X)$, path $X_0 \to v$ is equal to $LE((X^{\tau_v})^{\leftarrow})$

hitting time reversal.



Fact (Lawler): $LE((X^{\tau_v})^{\leftarrow}) \overset{d}{=} LE(X^{\tau_v})^{\leftarrow}$

NB: Does <u>not</u> hold pointwise.

$\Rightarrow$ Corollary.

3: Wilson's Algorithm

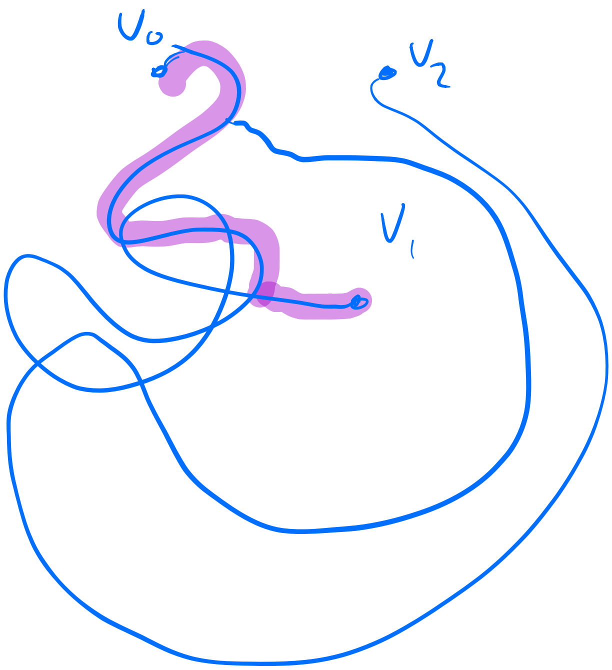
Enumerate $V \cong \{v_0, v_1, \dots, v_N\}$

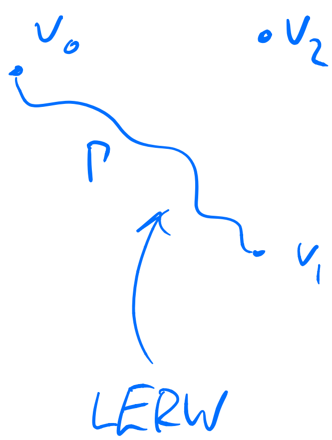
$T_0 = (\{v_0\}, \emptyset)$

Given T_n, let X^{n+1} be a random walk started at v_{n+1} & stopped when it hits vertex set of T_n.

T_{n+1} = Union of T_n with loop erasure of X^{n+1}.

$\boxed{\text{Thm : (Wilson) } T_N \sim \text{UST.}}$





Apply corollary again:

Conditional distr. of path from v_2 to Γ is LERW in G/Γ

Same thing as LERW in G from v_2 to Γ!

Markov property of UST:

Conditional distr. of $T \sim UST$ Given $A \subseteq T$, $T \cap B = \emptyset$

$$\overset{d}{=} A \cup \left(UST \text{ of } \frac{G - B}{A} \right)$$

delete B, contract A.

There is also a strong spatial Markov property:

Say that a random set $K \subseteq E$ defined on same probability space as T is a local set if

$$\{K \subseteq W\} \text{ is measurable wrt } \sigma(T|_W) \quad \forall W \subseteq E.$$

E.g. K = path connecting u and v.

Conditional on K and $T|_K$,

$$T \sim K_o \cup \left(\text{UST of } \frac{G - K_c}{K_o} \right)$$

open edges in K

closed edges in K.

Wilson's algorithm follows by earlier corollary + induction!

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§2: Infinite volume limits and the connectivity / disconnectivity transition.

$G = (V,E)$ ∞, locally finite, connected graph (e.g. $\mathbb{Z}^d$)

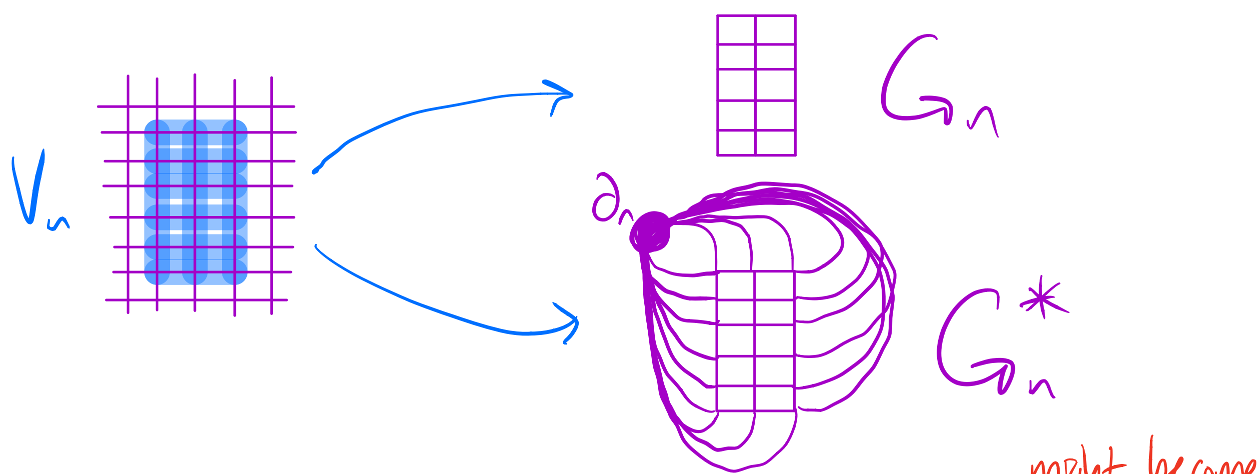
all degrees finite.

$(V_n)_{n\geq 1}$ exhaustion of V by finite connected sets

$$V_n \subseteq V_{n+1} \subseteq \cdots, \quad \bigcup_{n\geq 1} V_n = V$$

G_n : subgraph induced by V_n

G_n^* : Contract $V \setminus V_n$ to a single vertex ∂_n



<u>Free</u> Uniform Spanning Forest:

Weak limit of USTs of <u>G_n</u>

<u>Wired</u> Uniform Spanning Forest:

Weak limit of USTs of <u>G_n^*</u>

Both limits exist and are independent of the exhaustion (Pemantle '91)

$$\boxed{\text{Thm: For } \mathbb{Z}^d, \ \mathsf{FUSF} = \mathsf{WUSF}.}$$

Generally, $\mathsf{FUSF} \neq \mathsf{WUSF} \iff \exists$ nonconstant harmonic $h : V \to \mathbb{R}$ s.t. $\sum_{x \sim y} (h(x) - h(y))^2 < \infty$

Wilson's algorithm rooted at infinity

In particular, if G is an amenable transitive graph, then $\mathsf{FUSF} = \mathsf{WUSF}$

Benjamini, Lyons, Peres, Schramm

G Cayley Γ $\mathsf{FUSF} \neq \mathsf{WUSF}$ $\iff$ Γ has positive first ℓ^2 Betti number

G infinite, transient. $V = \{v_1, v_2, \ldots\}$

$\mathfrak{F}_0$ = empty forest with no vertices or edges

$$\ell^2(\Gamma) = \tfrac{1}{2}\big(\mathbb{E}\, d_{\mathsf{FUSF}}(id)\big) - \underbrace{\deg_{\mathsf{WUSF}}(id)}_{= 2}$$

Given $\mathfrak{F}_n$, let X^{n+1} be a random walk started at v_{n+1} and stopped when it hits vertex set of $\mathfrak{F}_n$, running forever if this never happens.

Let $\mathfrak{F}_{n+1} = \mathfrak{F}_n \cup LE(X^{n+1})$.

Then $\mathfrak{F} = \bigcup_{n \geq 1} \mathfrak{F}_n \sim \mathsf{WUSF}$

Connectivity / Disconnectivity

Key observation: When running Wilson's algorithm, a new component is formed exactly when X^{n+1} does not hit $\mathfrak{F}^n$.

$\boxed{\text{Thm (Pemantle '91) The USF of } \mathbb{Z}^d \text{ is connected when } d \leq 4 \text{ and has } \infty \text{ many components when } d > 4}$

$\boxed{\text{Thm (Erdös-Taylor '60s): Two simple random walks on } \mathbb{Z}^d \text{ intersect infinitely often a.s. when } d \leq 4, \text{ and at most finitely often a.s. when } d > 4.}$

High-dimensional part of Erdös–Taylor is easy:

X, Y random walks started at $x, y \in \mathbb{Z}^d$.

$$\mathbb{E}\,|\{n,m \geq 0 : X_n = Y_m\}|$$

$$= \sum_{n,m\geq 0} \sum_{z\in\mathbb{Z}^d} p_n(x,z)\, p_m(y,z)$$

$$= \sum_{n,m\geq 0} p_{n+m}(x,y) = \sum_{k\geq 0} (k+1)\, p_k(x,y)$$

Classical estimate: $p_k(x,y) \asymp \frac{1}{k^{d/2}} e^{-\Theta\left(\frac{\|x-y\|^2}{k}\right)}$

$$\rightsquigarrow \mathbb{E}\,|\{n,m \geq 0 : X_n = Y_m\}| \lesssim \|x-y\|^{-d+4}$$

when $d > 4$.

Also shows that this expectation is infinite when $d \leq 4$.

Low-dimensional part X, Y both start at 0

$$I_N = |\{0 \le n,m \le N : X_n = Y_m\}|$$

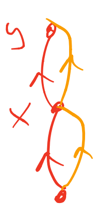
Calculation we just did shows that

$$\mathbb{E} I_N = \sum_{n,m=0}^{N} p_{n+m}(0,0) \asymp \begin{cases} N^{4-d/2} & d<4 \\ \log N & d=4 \\ 1 & d>4 \end{cases}$$

In particular, $\mathbb{E} I_N \xrightarrow[N\to\infty]{} \infty$ when $d \le 4$.

On the other hand, letting $g_N(x,y) = \sum_0^N p_n(x,y)$

$$\mathbb{E} I_N^2 \le \sum_{x,y} \left(g_N(0,x)\, g_N(x,y) + g_N(0,y)\, g_N(y,x) \right)^2$$



$$\le 2 \sum_{x,y} \left(g_N(0,x)^2\, g_N(x,y)^2 + g_N(0,y)^2\, g_N(y,x)^2 \right)$$

$$= 4 \sum g_N(0,x)^2\, g_N(x,y)^2 = 4 (\mathbb{E} I_N)^2$$

Paley-Zygmund:

$$\mathbb{P}(I_N \geq \tfrac{1}{2}\mathbb{E}I_N) \geq \frac{1}{4}\frac{(\mathbb{E}I_n)^2}{\mathbb{E}(I_n^2)} \geq \frac{1}{16}$$

Since $\mathbb{E}I_N \to \infty$ as $N \to \infty$, Fatou implies

$$\mathbb{P}(I = \infty) \geq \limsup \mathbb{P}(I_N \geq \tfrac{1}{2}\mathbb{E}I_N) \geq 1/16.$$

Hewitt-Savage 0-1 law implies $\{I = \infty\}$ is a 0-1 event $\square$.

High-d part of Pemantle:

Start k RWs all far away from each other. Then none of these walks intersect each other whp.

$\Rightarrow$ USF has at least k components whp.

Low d part:

transient

<u>Thm</u> (Lyons, Peres, Schramm) Let X, Y be random walks on some graph. Then

$$\mathbb{P}(Y \cap LE(X) \neq \emptyset) \geq \frac{1}{256}\, \mathbb{P}(Y \cap X \neq \emptyset)$$

Moreover, if X and Y intersect i.o. a.s. then so do Y and $LE(X)$.

<u>Exercise</u> Let $A \subseteq \mathbb{Z}^3$ be an ∞ connected set. Prove that SRW on $\mathbb{Z}^3$ intersects A infinitely often almost surely.

§3: One-endedness via interlacement
Aldous-Broder.

A tree is said to be <u>one-ended</u> if it does not contain a bi-infinite simple path.

Thm (BLPS) Every tree in the USF of $\mathbb{Z}^d$ is one-ended a.s. for every $d \geq 2$.

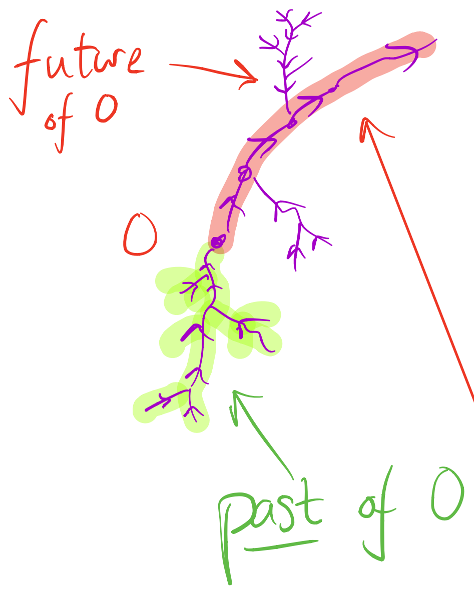
We'll prove the $d \geq 3$ case.

There are now many proofs available, applying at different levels of generality.

I will show an unpublished proof that leads naturally to the critical exponent calculations in high-d.

NB: <u>Oriented WUSF</u> : Orient the UST of G_n^* towards ∂_n before taking limit.

$\Longleftrightarrow$ Oriented version of Wilson's algorithm



Every vertex u has a unique ∞ oriented path emanating from it, called the *future* of u.

$v \in$ future of u $\iff$ $u \in$ *past* of v.

Future distributed as LERW by Wilson.

One-endedness $\iff$ Every vertex has finite past.

Critical exponents: E.g.

$$\mathbb{P}(|\text{Past}| \geq n) \approx n^{-1/8}$$

$$\mathbb{P}(\text{Past reaches distance } r) \approx r^{-1/3}$$

(We'll have much more to say about this later.)

§3.1 The Interlacement Aldous-Broder algorithm

Random interlacements:

- Introduced by Sznitman
- "Poissonian soup of doubly-infinite random walk trajectories".

Let G be an ∞, locally finite graph.

For each $-\infty \leq n \leq m \leq +\infty$, let

$W(n,m)$ = set of transient paths in G indexed by $[n,m]$.

visits each vertex at most finitely often

$$W = \bigcup_{-\infty \leq n \leq m \leq \infty} W(n,m)$$ set of all transient paths.

The topology on W:

For each $K \subseteq V$ finite, let $W_K \subseteq W$ be set of paths hitting K.

For each $w \in W_K$, let w_K be the portion of w between first and last visits to K.

The topology on W is generated by the subbasis of open sets

$$\left\{ \{ w \in W : w \in W_K,\ w_K = w'_K \} : \begin{array}{l} K \subseteq V \text{ finite}, \\ w' \in W_K \end{array} \right\}$$

(This is ~~not~~ stronger than the product topology!)

This topology makes W into a Polish space.

- First and last hitting times, local times at vertices, evaluation at a time all continuous.

$W^* = W/\sim$ where $w_1 \sim w_2$ if $\exists$ k s.t. $w_1(i) = w_2(i+k)$ $\forall i$

W^* is equipped with the quotient topology, which makes it a Polish space.

Elements of W^* are called <u>trajectories</u>

$\boxed{\text{The random interlacement process is a Poisson point process on } W^* \times \underset{\text{time}}{\mathbb{R}}.}$

The intensity measure:

- For each $w \in W(n,m)$, define $w^{\leftarrow} \in W(-m,-n)$ to be the reversal of w, $w^{\leftarrow}(i) = w(-i)$.

For each $K \subseteq V$ finite, define a measure Q_K on W_K by

arbitrary Borel subsets of W $\qquad \in V$

$$Q_K\left(w|_{(-\infty,0]} \in A,\ w|_{[0,\infty)} \in B,\ w(0) = u\right)$$

$$= \mathbb{1}(u \in K)\deg(u)\,\mathbb{P}_u\left(X^{\leftarrow} \in A,\ \tau_K^+ = \infty\right)\mathbb{P}_u(X \in B)$$

law of SRW

first positive hitting time

$\forall\, u \in V$ and $A, B \subseteq W$ Borel.

$\pi : W \longrightarrow W^*$ projection

$W^*_K = \pi(W_K)$ = trajectories visiting K.

<u>Thm</u> (Sznitman/Texeira) G ∞, locally finite, transient. Then there exists a unique locally finite measure Q^* on W^* such that

$$Q^*(A \cap W^*_K) = Q_K(\pi^{-1}(A))$$

for every $A \subseteq W^*$ and $K \subseteq V$ finite.

Q^* : interlacement intensity measure.

Def^n: The random interlacement process on G is the Poisson point process on $W^* \times \mathbb{R}$ with intensity

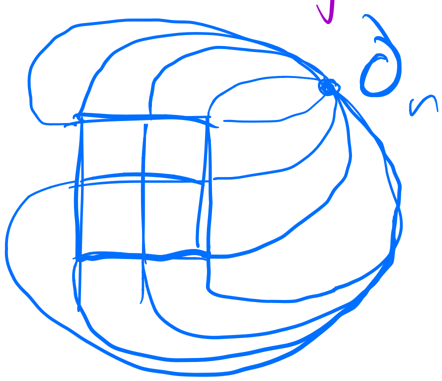
$Q^* \otimes$ Lebesgue.

The limiting construction

$G = (V, E)$ ∞, locally finite, transient.

$V = \bigcup V_n$ exhaustion by finite sets,

G_n^* defined as before w/ boundary vertex ∂_n



- Take an intensity $\deg(\partial_n)$ Poisson point process on $\mathbb{R}$.
- For each time t in the process, let W_t be a random walk excursion from ∂_n to itself, which we consider as an element of W^*.

- Get a random set

$$\mathcal{I}_n = \{ (W_{t_i}, t) \} \subseteq W^* \times \mathbb{R}$$

$\boxed{\text{\underline{Prop} } \mathcal{I}_n \text{ converges weakly to the random interlacement process on } G.}$

In other words:

- Take doubly ∞ SRW on G_n^*.
- Break up into excursions from the boundary.
- Apply Poissonian time change. Scale time in such a way that we get a non-degenerate limit.

Interlacements $\longleftrightarrow$ local picture of SRW on the time scale that it covers the graph.

The interlacement Aldous-Broder algorithm

$G=(V,E)$ ∞, locally finite, ~~transitive~~ transient graph.

$\mathcal{I}$ random interlacement process on G.

For each $v \in V$ and $t \in \mathbb{R}$,

$\sigma_t(v)$ = First time after time t that v is hit by a trajectory $W_{\sigma_t(v)}$

$e_t(v)$ = Oriented edge traversed by $W_{\sigma_t(v)}$ as it enters v for the first time.

Thm (H. 2015)

$$AB_t(\mathcal{I}) = \{ e_t(v)^{\leftarrow} : v \in V \}$$

is distributed as the oriented wired uniform spanning forest for each $t \in \mathbb{R}$

$AB_t(\mathcal{I}_n) \sim \mathrm{UST}$ of G_n^*

by usual Aldous-Broder

$\rightsquigarrow$ STP $AB_t(\mathcal{I}_n) \xrightarrow[n\to\infty]{\text{weakly}} AB_t(\mathcal{I})$.

Although $AB_t : \mathcal{W}^* \times \mathbb{R} \to \{0,1\}^{\vec{E}}$ is not continuous, its has enough continuity properties to prove this by standard arguments (portmanteau theorem).

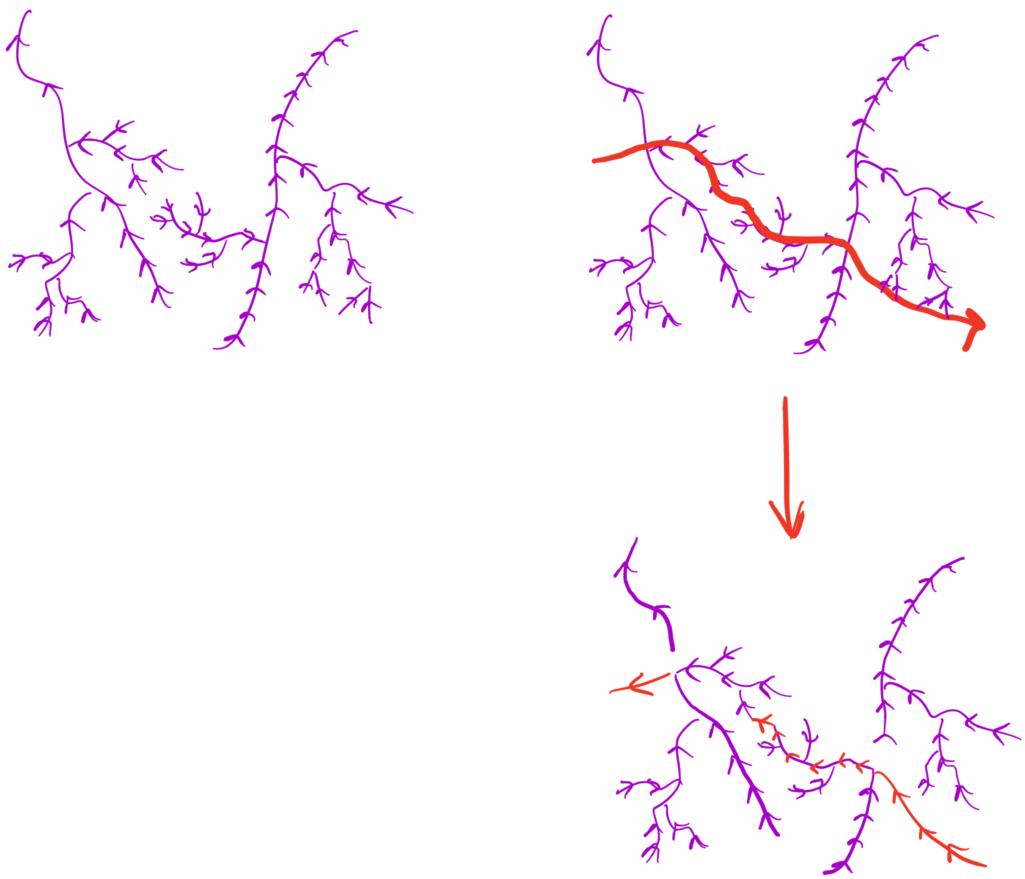
$(AB_t(\mathcal{I}))_{t\in\mathbb{R}}$ is a stationary ergodic, stochastically continuous Markov process with stationary measure OWUSF.

This dynamical viewpoint will be of central importance to our analysis of the WUSF in high dimensions.

The past of a vertex evolves in a nice way under the dynamics:

$\tilde{\mathcal{F}}_t = AB_t(\mathcal{I})$ $\quad P_t(v) =$ past of v in $\tilde{\mathcal{F}}_t$.

As t decreases, new trajectories arrive and overwrite what was there previously

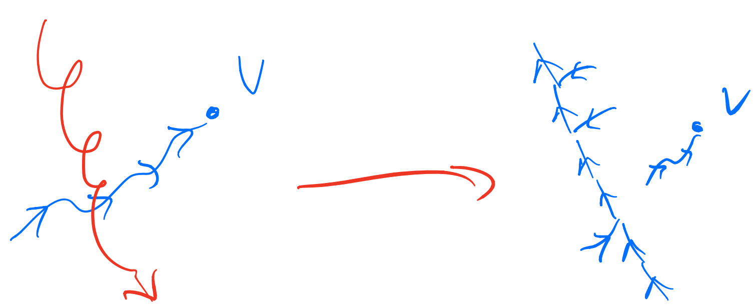


$\mathcal{I}_{[s,t)}$ vertices hit in $[s,t)$

$\boxed{\text{Lemma: If } v \notin \mathcal{I}_{[s,t)} \text{ then } P_s(v) = \text{component of } v \text{ in } P_t(v) \setminus \mathcal{I}_{[s,t)}.}$

- Let $u \in V$ and let $F_t(u) = (F_t(u)_0 = u, F_t(u)_1, \dots)$ be the future of u in $\mathcal{F}_t$
- $\sigma_t(F_t(u)_i)$ is decreasing in i.
- $F_s(u)$ and $F_t(u)$ agree until they reach a vertex with $\sigma_s(v) < t$.

After this step, every element of $F_s(u)$ has $\sigma_s < t$.



Interlacement hitting probabilities & the capacity.

Let $K \subseteq V$ be finite. The set of times in which K is visited by an interlacement trajectory is a Poisson process with intensity

$$Q^*(W_K^*) = Q_K(W)$$

$$= \sum_{u \in K} \deg(u)\, \mathbb{P}_u(\tau_K^+ = \infty)$$

This quantity is known as the <u>capacity</u> or <u>conductance to infinity</u> of K

$$Q^*(W_K^*) = \mathrm{Cap}(K) = \mathcal{C}_{\mathrm{eff}}(K \to \infty)$$

The theory of electrical networks gives many ways to compute / estimate this quantity.

E.g. "Newtonian capacity" formulation (Jain & Orey '73)

The Greens function $g(u,v) = \sum_{n \geq 0} p_n(u,v)$

$$\mathrm{Cap}(K)^{-1} = \inf\left\{ \sum_{u,v \in K} \frac{g(u,v)}{\deg(v)} \mu(u)\mu(v) : \mu \text{ a prob. meas. on } K \right\}$$

For $\mathbb{Z}^d$, $g(x,y) \asymp \|x-y\|^{-d+2}$

$$\mathrm{Cap}(K)^{-1} \lesssim \frac{1}{|K|^2} \sum_{x,y \in K} \|x-y\|^{-d+2}$$

Ball is worst case

$$\rightsquigarrow \boxed{\mathrm{Cap}(K) \gtrsim |K|^{d-2/d}}$$

(cf. isoperimetric inequality $|\partial K| \gtrsim |K|^{\frac{d-1}{d}}$)

Quantitative one-endedness on $\mathbb{Z}^d$, $d \geq 3$.

$P_t(v,n) =$ vertices in the past of v with intrinsic distance at most n

↑ the graph metric on $\tilde{\mathfrak{F}}_t$

$$\partial P_t(v,n) = P_t(v,n) \setminus P_t(v,n-1).$$

Theorem Let $d \geq 3$ and consider the USF of $\mathbb{Z}^d$. Then

$$\mathbb{P}(\partial P_0(0,n) \neq \emptyset) \leq C \frac{\log n}{n^{(d-2)/d}}.$$

Corollary: Every tree is one-ended a.s.

Similar theorem due to Lyons, Morris & Schramm '03.

Subpotimal result with simple proof.

The mass-transport principle:

If $F: \mathbb{Z}^d \times \mathbb{Z}^d \to [0,\infty)$ satisfies

$F(x,y) = F(x+z, y+z)$ $\forall x,y,z$ then

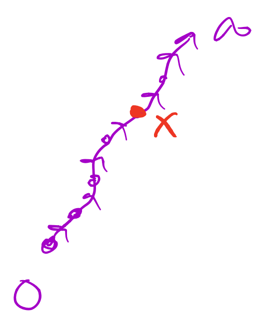
$$\boxed{\sum_x F(0,x) = \sum_x F(x,0)}$$

E.g. $\mathbb{E}|\partial P_0(0,n)| = 1$ for all n:

$$F(x,y) = \mathbb{P}(y \in \partial P_0(x,n))$$

$$\sum F(0,x) = \mathbb{E}|\partial P_0(0,n)|$$

$$\sum F(x,0) = \mathbb{E}|\{x : 0 \in \partial P_0(x,n)\}| = 1.$$

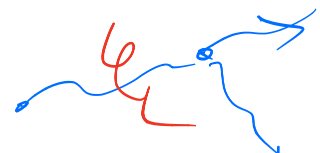


Only one such x n steps in the future of 0.

Proof of Theorem

$$\mathbb{P}(\partial P_0(0,n) \neq \emptyset) \le \overbrace{\mathbb{P}(\sigma_0(0) \le \varepsilon)}^{O(\varepsilon)} + \mathbb{P}(\sigma_0(0) > \varepsilon,\ \partial P_0(0,n) \neq \emptyset)$$

Second term



$$\le \mathbb{P}\left(\begin{array}{l} \partial P_\varepsilon(0,n) \neq \emptyset \text{ and } \exists \text{ a geodesic} \\ 0 \xrightarrow{\varepsilon} \partial P_\varepsilon(0,n) \text{ that is not in } \tilde{I}_{[0,\varepsilon)} \end{array} \right)$$

(By lemma)

$$= \mathbb{P}\left(\begin{array}{l} \partial P_0(0,n) \neq \emptyset \text{ and } \exists \text{ a geodesic} \\ 0 \longrightarrow \partial P_0(0,n) \text{ that is not in } \tilde{I}_{[-\varepsilon,0)} \end{array} \right)$$

$$\le \mathbb{E}\, \# \left\{ u : \begin{array}{l} u \in \partial P_0(0,n), \text{ path from } u \text{ to} \\ 0 \text{ not in } \tilde{I}_{[-\varepsilon,0)} \end{array} \right\}$$

$$\le \mathbb{E} \sum_{u \in \partial P_0(0,n)} e^{-\varepsilon \operatorname{Cap}(0 \rightsquigarrow u)}$$

$$\leq e^{-c\varepsilon n^{(d-2)/d}} \underbrace{\mathbb{E}\#\{u : u \in \partial P_0(0,n)\}}_{=1}$$

↑ $\mathcal{F}_0$ and $\mathcal{I}_{[-\varepsilon,0)}$ independent, hitting controlled by capacity.

So $\mathbb{P}(\partial P_0(0,n) \neq \emptyset) \leq C\varepsilon + e^{-c\varepsilon n^{\frac{d-2}{d}}}$

Take $\varepsilon = C' n^{-d-2/d} \log n$ $\quad C'$ large

$\Rightarrow$ claim $\square$

This proof can be made to work for e.g. all transient transitive graphs, recovering a result of Lyons, Morris, and Schramm.

To improve this inequality we will need a better understanding of the capacity of LERW

$$\mathbb{P}(\text{Past} \cap \mathbb{Z}^d \setminus [-r,r]^d \neq \emptyset) \asymp \frac{1}{r^2}$$

§4: Critical exponents for $d \geq 4$.

4.1: The big picture.

General picture for critical stat mech models:

- Tails of interesting random variables governed by power laws. Exponent should depend on dimension but not choice of lattice

E.g. critical percolation

$$\mathbb{P}_{p_c}(|\text{Cluster of origin}| \geq n) \approx n^{-1/\delta + o(1)}$$

- There is an "upper critical" dimension above which the exponents stabilize at their "mean-field" values.

"Mean-field" $\longleftrightarrow$ same behaviour as on complete graph / binary tree.

terminology comes from field theories like Ising model.

Mean-field behaviour means that having a local interaction

$$e^{-\beta \sum_{x \sim y} \sigma_x \sigma_y} \text{ behaves similarly to } e^{-\beta' \sum_x \sigma_x \frac{1}{|V|} \sum \sigma_y}$$

For percolation, $d_c = 6$.

Conjecturally:

$$\mathbb{P}_{p_c}(|K_0| \geq n) \approx \begin{cases} n^{-5/91} & d=2 \\ n^{-1/\delta(d)} & 3 \leq d \leq 5 \\ n^{-1/2} \log^{\gamma} n & d=6 \\ n^{-1/2} & d>6 \end{cases}$$

$d=2$ understood for site perc on triangular lattice (Smirnov, Lawler, Schramm, Werner, Kesten)

$d >> 6$ Hara - Slade '90s

$3 \leq d \leq 6$ totally open.

For the UST, $d_c = 4$.

Theorem (H. 2018) If $d \geq 5$

$$\mathbb{P}(\partial P_0(0,n) \neq \emptyset) \asymp 1/n.$$

Theorem (H. & Sousi 2020+) When $d=4$,

$$\mathbb{P}(\partial P_0(0,n) \neq \emptyset) \asymp \frac{(\log n)^{1/3}}{n}.$$

- Can derive other related exponents once these are known.
- $d=4$ relies on analysis of 4d LERW due to Lawler.
- Related results on scaling limit of OST of torus due to

  Peres & Revelle 2004 $d \geq 5$
  Schweinsberg 2009 $d=4$

In both cases there is convergence to the CRT, but the scaling is different in 4d.

- Ideas from our high-d proof have also been used to analyze the diameter of the UST on finite high-d graphs by Michaeli, Nachmias & Shalev (2019).

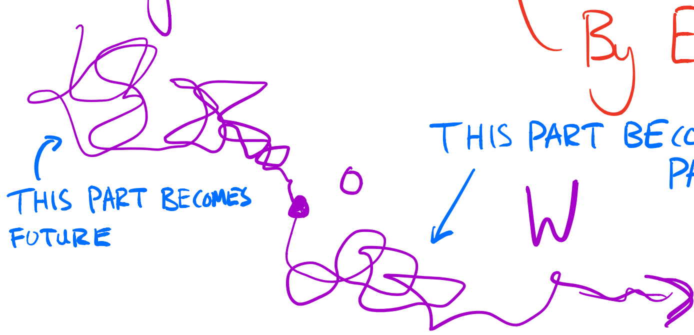
Lower bounds in high dimensions

Take $d \geq 5$ and let $\mathcal{I}$ be the interlacement process on $\mathbb{Z}^d$.

$$\boxed{\text{Thm: } \mathbb{P}(\partial P_0(0,n) \neq \emptyset) \geq \frac{c}{n}.}$$

- 0 is hit by a unique trajectory in $[0, \frac{c}{n}]$ with probability $\asymp 1/n$.
- With constant probability, the parts of the trajectory before and after hitting 0 are disjoint!

By Erdös-Taylor. Needs $d \geq 5$!



- If this were the only trajectory, the past of O would be infinite!

— If we apply Aldous-Broder to W we get a tree in which the past of O is infinite.

Let η be an infinite simple path in the past of O in $AB(W)$

One can prove that η is unique and is distributed as a conditioned LERW, but we won't need this for now.

- The forest $\mathcal{F}_0$ includes the part of η up until the first vertex that is visited by a trajectory that arrived before W.

- By the splitting property of Poisson processes, the first n steps of η

are <u>not</u> hit by any earlier trajectory with prob. at least

$$e^{-\frac{C}{n}\mathrm{Cap}(\text{First } n \text{ steps of } \eta)} \geq c' > 0.$$

$$\mathrm{Cap}(A) = \sum_{a\in A} \deg(a)\, \mathbb{P}_a(\tau_A^+ = \infty)$$
$$\preccurlyeq |A|$$

So we've shown that in high dimensions

$$\mathbb{P}(\partial P_0(0,n) \neq \emptyset \mid 0 \text{ hit in } [0, \tfrac{1}{n}]) \geq c > 0.$$

and hence

$$\mathbb{P}(\partial P_0(0,n) \neq \emptyset) \asymp 1/n \quad \text{as claimed.} \quad \square$$

Same strategy gives correct answer in $\mathbb{Z}^3$ & $\mathbb{Z}^4$, but is more difficult to implement rigorously.

Two competing effects

- Capacity of n step LERW grows sublinearly
- Probability two random walks don't intersect decays.

In 4d:

Capacity of n step LERW $\asymp \frac{n}{(\log n)^{2/3}}$

Lawler:

$\mathbb{P}(\text{SRW avoids an } n \text{ step LERW}) \asymp 1/(\log n)^{1/3}$

$\rightsquigarrow$ Take $\varepsilon = \frac{(\log n)^{2/3}}{n}$, get

$$\mathbb{P}(\partial P_0(0,n) \neq \emptyset) \gtrsim \frac{(\log n)^{2/3}}{n} \frac{1}{(\log n)^{1/3}}$$

$$= \frac{(\log n)^{1/3}}{n}$$

To do this properly, one needs to know not just the 'typical' order of the capacity, but to have good enough concentration that the large capacity "bad event" has negligible probability

Upper bounds when $d \geq 5$.

Want to prove $Q(n) := \mathbb{P}(\partial P_0(0,n) \neq \emptyset) \leq \frac{C}{n}$.

Suffices to prove inductive inequality

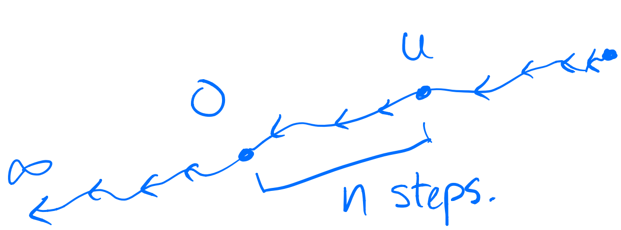
$$Q(2n) \leq \frac{C}{n} + \frac{1}{4} Q(n)$$

$< 1/2$.

Similarly to last time

$$Q(2n) \leq \underbrace{\mathbb{P}(\sigma_0(0) \leq \varepsilon)}_{= O(\varepsilon)} +$$

$$\mathbb{E}\#\left\{ u : \begin{array}{c} u \in \partial P_\varepsilon(0,n), \text{ path from } \\ u \text{ to } v \text{ in } \hat{\mathfrak{F}}_\varepsilon \text{ not hit in } [0,\varepsilon), \\ \text{AND } \partial P_\varepsilon(u,n) \neq \emptyset \end{array} \right\}$$



Bound by "High Capacity" and "Low Capacity" terms.

$$\mathbb{E}\#\left\{u : \begin{array}{c} u \in \partial P_\varepsilon(0,n),\ \text{path from} \\ u \text{ to } v \text{ in } \mathcal{J}_\varepsilon \text{ not hit in } [0,\varepsilon), \\ \text{AND } \partial P_\varepsilon(u,n) \neq \emptyset \end{array}\right\}$$

$$\leq e^{-\varepsilon\delta n} \underbrace{\mathbb{E}\#\{u : u \in \partial P_\varepsilon(0,n) \text{ and } \partial P_\varepsilon(u,n) \neq \emptyset\}}_{= Q(n) \text{ by mass transport!}}$$

$$+ \underbrace{\mathbb{E}\#\left\{u : \begin{array}{c} u \in \partial P_\varepsilon(0,n),\ \partial P_\varepsilon(u,n) \neq \emptyset, \\ \underline{\text{and}}\ \mathrm{Cap}(0 \rightsquigarrow u) \leq \delta n \end{array}\right\}}_{(*)}$$

$$(*) = \mathbb{P}\left(\begin{array}{c} \partial P_0(0,n) \neq \emptyset \text{ and first} \\ n \text{ steps of future have } \mathrm{Cap} \leq \delta n \end{array} \right)$$

Suppose we can show that if δ is sufficiently small then

$$(*) \leq \frac{1}{8} Q(n).$$

Then we get

$$Q(2n) \leq C\varepsilon + e^{-\varepsilon \delta n} Q(n) + \frac{1}{8} Q(n)$$

$\rightsquigarrow$ Take δ small, $\varepsilon = \dfrac{\log 8}{\delta n}$

$\rightsquigarrow$ Done.

The capacity of SRW & LERW

$$\mathrm{Cap}(A) = \sum \deg(v)\, \mathbb{P}_v(\tau_A^+ = \infty)$$

X^n first n steps of random walk on $\mathbb{Z}^d$.

How does $\mathrm{Cap}(X^n)$ grow?

Related to non-intersection probabilities.

$$\mathbb{E}\,\mathrm{Cap}(X^n) =$$

$$2d \sum_{m=0}^{n} \mathbb{P}\left(\begin{array}{c} Y \text{ does not hit } (X^m \cup Z^{n-m}) \\ \text{after time } 0 \end{array} \right)$$

Erdös-Taylar $\Rightarrow$

$\mathbb{E}\, Cap(X^n) \asymp n$ when $d \geq 5$.

Lawler:

$\mathbb{E}\, Cap(X^n) \asymp \frac{n}{\log n}$ when $d = 4$

Asselah, Shapira, Sousi : Good concentration in both cases.

Easy argument giving lower bound of correct order:

$$Cap(A)^{-1} = \inf \left\{ \sum_{u,v \in A} \frac{\xi(u,v)}{\deg(v)} \mu(u)\mu(v) \; : \; \begin{matrix}\mu \text{ prob} \\ \text{meas on} \\ A\end{matrix} \right\}$$

$$\preceq \frac{1}{|A|^2} \sum_{u,v \in A} \xi(u,v)$$

cn whp when $d \geq 3$.

$$\rightsquigarrow Cap(X^n) \geq \frac{|\{X_m : 0 \leq m \leq n\}|^2}{\sum_{i,j=0}^{n} \xi(X_i, X_j)}$$

One can compute that

$$\mathbb{E}\sum_{i,j=0}^{n} g(X_i, X_j)$$

$$\leq 2(n+1)\sum_{m=0}^{n}(m+1)\,p_m(0,0) + 2(n+1)^2\sum_{m=n+1}^{\infty} p_m(0,0)$$

$$\asymp \begin{cases} n & d \geq 5 \\ n\log n & d = 4 \\ n^{3/2} & d = 3 \end{cases}$$

Similar to proof of Erdős–Taylor

Gives lower bound of the correct order for the capacity in each case.

What about LERW?

Earlier this theorem was mentioned:

transient

Thm (Lyons, Peres, Schramm) Let X, Y be random walks on some graph. Then

$$\mathbb{P}(Y \cap LE(X) \neq \emptyset) \geq \frac{1}{256} \mathbb{P}(Y \cap X \neq \emptyset)$$

Moreover, if X and Y intersect i.o. a.s. then so do Y and LE(X).

With Perla, we show that essentially the same proof gives the following:

Thm (H & Sousi) Let G be a transient graph, and let X be a random walk on G. Then

$$\mathbb{E}\,\mathrm{Cap}(LE(X^n)) \geq \frac{1}{256} \mathbb{E}\,\mathrm{Cap}(X^n) \quad \forall n \geq 1$$

In 4d, Lawler shows that

$$|LE(X^n)| \asymp \frac{n}{(\log n)^{1/3}} \quad \text{whp,}$$

So Cap(First n steps of LERW)

$$\approx \mathrm{Cap}\left(LE\left(X^{n(\log n)^{1/3}}\right)\right) \asymp \frac{(\log n)^{2/3}}{n}.$$

This statement is only in expectation. We show that it holds ~~with~~ high probability also. (There is concentration)

In high dimensions it is a much easier matter to get that the capacity of the LERW is linear with high probability.

There is still a problem:

(*) $\mathbb{P}\left(\begin{array}{l} \partial P_0(0,n) \neq \emptyset \text{ and first} \\ n \text{ steps of future have } Cap \leq \delta n \end{array} \right)$

These events aren't independent!

If they were, we could take δ suff small and get

$$(*) \leq \frac{1}{8} Q(n) \quad \text{as required.}$$

Not a problem if

$$\mathbb{P}\left(Cap\,(n \text{ step LERW}) \leq \delta n \right) \ll 1/n$$

This should work when $d >> 4$, might not for $d=5$.

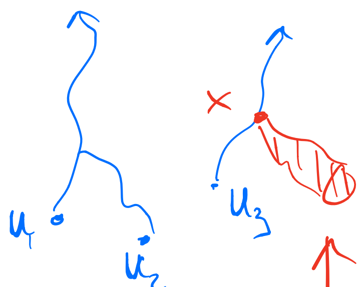
The v-wired uniform spanning forest

Introduced by Járai & Redig.

G_n^{*v} : Identify v and ∂_n in G_n^*

v-WUSF is weak limit of USTs of G_n^{*v} oriented towards $\partial_n = v$.

Stochastic domination property (Lyons, Morris, Schramm)



Condition on futures of $u_1, \ldots, u_k$ in WUSF or v-WUSF

Conditional distribution of the part of the past of x that does not belong to the revealed forest is stochastically dominated by the component of x in the x-WUSF.

Plays an analogous role to BK inequality in percolation.

In particular, if we define

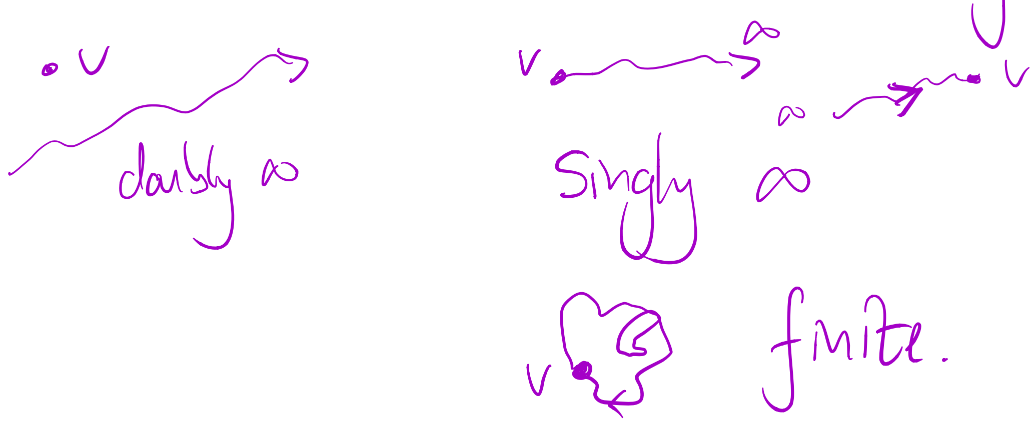
$$Q_o(n) = \mathbb{P}(o\text{-WUSF has intrinsic radius at least } n)$$

Then $Q(n) \le Q_o(n) \quad \forall n$.

There are analogues of Wilson & interlacement Aldous-Broder for the v-WUSF.

Wilson same as before but take $\tilde{F}_0 = (\{v\}, \emptyset)$

"v-wired interlacement" includes trajectories



Otherwise similar to before.

$$Q_0(2n) \le \mathbb{P}(\sigma_0(0) \le \varepsilon) + $$ still $O(\varepsilon)$

$$\mathbb{E}\#\left\{u : \begin{array}{c} u \in \partial P_\varepsilon(0,n), \text{ path from } \\ u \text{ to } v \text{ in } \tilde{\mathfrak{F}}_\varepsilon \text{ not hit in } [0,\varepsilon), \\ \text{AND } \partial P_\varepsilon(u,n) \neq \emptyset \end{array}\right\}$$

All forests, interlacements are now 0-wired

Split into high capacity and low capacity terms as before.

$$\le e^{-\varepsilon\delta n} \underbrace{\mathbb{E}\#\{u : u \in \partial P_\varepsilon(0,n) \text{ and } \partial P_\varepsilon(u,n) \neq \emptyset\}}_{\le Q_0(n)\, \mathbb{E}\#\{u : u \in \partial P_\varepsilon(0,n)\}}$$

by stochastic domination.

$$+\ \mathbb{E}\#\{u : u \in \partial P_\varepsilon(0,n),\ \partial P_\varepsilon(u,n) \neq \emptyset, \ \underline{\text{and}}\ \mathrm{Cap}(0 \rightsquigarrow u) \le \delta n\}$$

$$\le Q_0(n)\, \mathbb{E}\#\{u : u \in \partial P_0(0,n) \text{ and } \mathrm{Cap}(0 \rightsquigarrow u) \le \delta n\}$$

New problem: no mass-transport!

Luckily, fairly simple analysis gives

$$\mathbb{E}\#\{u : u \in \partial P_0(0,n)\} \le C$$

$$\mathbb{E}\#\{u : u \in \partial P_0(0,n),\ \mathrm{Cap} \le \delta n\} \le \varepsilon_\delta$$

in high dim.

$\varepsilon_\delta \to 0$ as $\delta \downarrow 0$.

<u>Problem</u>: For $d \le 4$,

$$Q_0(n) >> Q(n).$$

This makes things much harder!

E.g. in 4d $\quad Q(n) \asymp \frac{(\log n)^{1/3}}{n}$

$$Q_0(n) \asymp \frac{(\log n)^{2/3}}{n}.$$

In 3d, the powers are different.

In 2d, $Q_0(n)$ is bounded below!!

Connection to sandpile model (Járai & Redig)

$$\text{Past of } 0 \text{ in WUSF} \le \text{Sandpile Avalanche} \le \text{Component of } 0 \text{ in } 0\text{-WUSF}$$

In high dim these have the same asymptotics, so we can obtain good understanding of the sandpile.

For $d \le 4$ there is a gap!

Open problem: Is

$$Q(n) \geq \frac{c}{n}$$ in every dimension?

Would give $\beta \geq 3/2$ where β is the dimension of 3d LERW.

Numerically $\beta \approx 1.62400\ldots$ (Wilson)

Rigorously $1 < \beta \leq 5/3$ (Lawler).

It is known that $Q_0(n) \geq \frac{c}{n} \; \forall n$