Critical percolation

Gady Kozma

Online Open Probability School, 2020

Percolation - definitions

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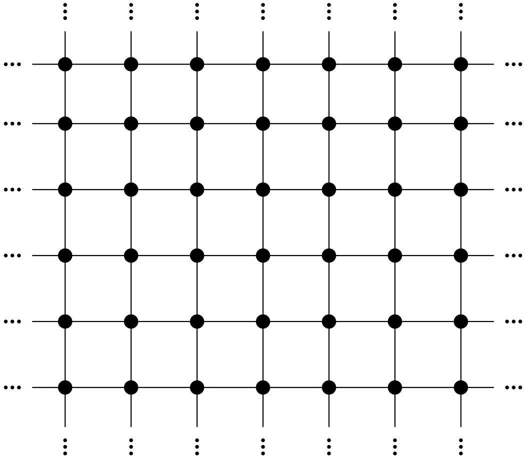

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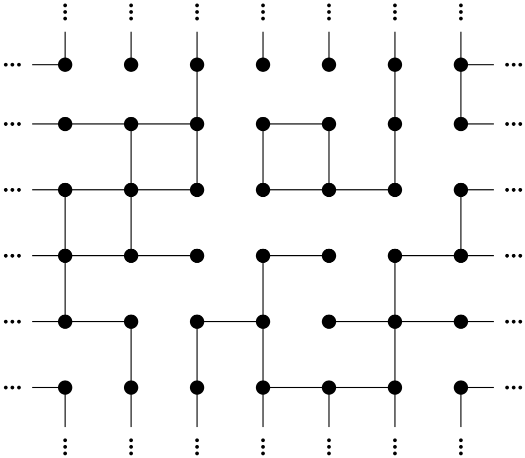

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This minicourse will focus on recent advances around this problem, with particular emphasis on the growing understanding of the importance of the Aizenman-Kesten-Newman argument. (but we will only get to it in the second hour)

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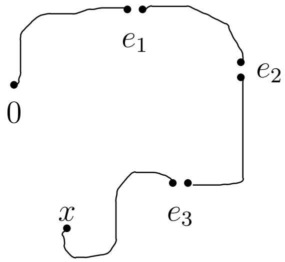

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$\chi = \mathbb{E}_p(|\mathscr{C}(0)|)$, $\varepsilon < 1/4d\chi$, $E_{x,e_1,\ldots,e_n}$ is the event that $\exists \gamma_i$ from e_{i-1}^+ to e_i^-, disjoint, and all e_i are sprinkled.

$$\mathbb{P}_{p+\varepsilon}(0 \leftrightarrow x) \leq \sum_{n=0}^{\infty} \sum_{e_1,\ldots,e_n} \mathbb{P}(E_{x,e_1,\ldots,e_n}).$$

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By the BK inequality

$$\leq \sum_{n=0}^{\infty} \sum_{e_1,\ldots,e_n} \mathbb{P}_p(0 \leftrightarrow e_1^-)\mathbb{P}_p(e_1^+ \leftrightarrow e_2^-) \cdots \mathbb{P}(e_n^+ \leftrightarrow x)\varepsilon^n$$

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Summing over all x gives

$$\chi(p+\varepsilon) \leq \sum_{n=0}^{\infty} \varepsilon^n \sum_{x,e_1,\dots,e_n} \mathbb{P}_p(0 \leftrightarrow e_1^-)\mathbb{P}_p(e_1^+ \leftrightarrow e_2^-)\cdots\mathbb{P}_p(e_n^+ \leftrightarrow x).$$

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Summing over x gives one $\chi(p)$ term which we can take out of the sum

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e_n^+ has $2d$ possibilities. Summing over e_n^- gives another χ term. Taking both out of the sum gives

$$= \sum_{n=0}^{\infty} \varepsilon^n \cdot 2d\chi(p)^2 \sum_{e_1, \ldots, e_{n-1}} \mathbb{P}_p(0 \leftrightarrow e_1^-) \cdots \mathbb{P}_p(e_{n-2}^+ \leftrightarrow e_{n-1}^-).$$

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$\chi(p) = \mathbb{E}_p(|\mathscr{C}(0)|)$, $\varepsilon < 1/4d\chi(p)$,

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This is sharp on a tree but not in general.

For a set $S \subset \mathbb{Z}^d$ denote by ∂S the set of $x \in S$ with a neighbour $y \notin S$.

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Let $S \subset \mathbb{Z}^d$ be some finite set containing 0*. Then*

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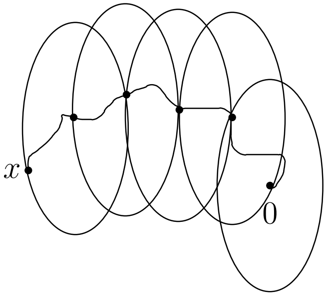

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And we have $n \geq r|x|$ for some number $r > 0$ that depends on S. A calculation similar to the previous proof shows that

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If the value in the parenthesis is smaller than 1 then $\mathbb{P}(0 \leftrightarrow x)$ decays exponentially in $|x|$, contradicting the previous theorem. □

Theorem

Let $S \subset \mathbb{Z}^d$ *be some finite set containing* 0*. Then* $\sum_{x \in \partial S} \mathbb{P}_{p_c}(0 \overset{S}{\longleftrightarrow} x) \geq 1$.

A full proof can be found in H. Duminil-Copin and V. Tassion, *A new proof of the sharpness of the phase transition for Bernoulli percolation on* $\mathbb{Z}^d$, L'Enseignement Mathématique, 62(1/2) (2016), 199-206.

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For any $p < p_c$ $\chi(p) < \infty$.

(recall that $\chi(p) = \mathbb{E}_p(|\mathscr{C}(0)|)$ and that what we proved before is $\chi(p_c) = \infty$).

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Two applications:

Lemma (K-Nachmias, 2011)

For any $x \in \partial\Lambda_n$, $\Lambda_n := [-n, n]^d$,

$$\mathbb{P}_{p_c}(0 \stackrel{\Lambda_n}{\longleftrightarrow} x) \geq c \exp(-C \log^2 n).$$

Lemma (Cerf, 2015)

For any $x, y \in \Lambda_n$,

$$\mathbb{P}_{p_c}(x \stackrel{\Lambda_{2n}}{\longleftrightarrow} y) \geq cn^{-C}.$$

All constants c and C might depend on the dimension.

Lemma (Cerf, 2015)

For any $x, y \in \Lambda_n$, $\mathbb{P}_{p_c}(x \xleftrightarrow{\Lambda_{2n}} y) \geq cn^{-C}$.

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Assume first that $x - y = (2k, 0, \ldots, 0)$, $k \leq n$.

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By rotation and reflection symmetry we may assume z is in some face of Λ_k, for example $z_1 = k$.

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$$\mathbb{P}(x \xleftrightarrow{x+\Lambda_k} x + z), \mathbb{P}(y \xleftrightarrow{y+\Lambda_k} y + \overline{z}) \geq \frac{c}{k^{d-1}}.$$

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But $x + z = y + \overline{z}$!

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By FKG

$$\mathbb{P}(x \overset{\Lambda_{2n}}{\longleftrightarrow} y) \geq \mathbb{P}(x \overset{\Lambda_{2n}}{\longleftrightarrow} x + z, y \overset{\Lambda_{2n}}{\longleftrightarrow} y + \overline{z}) \geq \frac{c}{k^{2d-2}}.$$

Proving the lemma in this case.

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Assume first that $x - y = (2k, 0, \ldots, 0)$, $k \leq n$. Then $\mathbb{P}(x \xleftrightarrow{\Lambda_{2n}} y) \geq ck^{2-2d} \geq cn^{2-2d}$.

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$$x = x_0, \ldots, x_d = y$$

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$$\begin{aligned}\mathbb{P}(x \xleftrightarrow{\Lambda_{2n}} y) &\geq \mathbb{P}(x_0 \xleftrightarrow{\Lambda_{2n}} x_1, x_1 \xleftrightarrow{\Lambda_{2n}} x_2, \dots, x_{d-1} \xleftrightarrow{\Lambda_{2n}} x_d) \\ &\geq \prod_{i=1}^{d} \mathbb{P}(x_{i-1} \xleftrightarrow{\Lambda_{2n}} x_i) \geq \frac{c}{n^{2d^2-2d}}.\end{aligned}$$

□

Lemma (Cerf, 2015)

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skip to diagram

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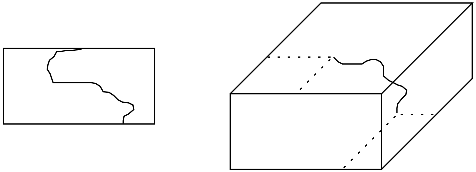
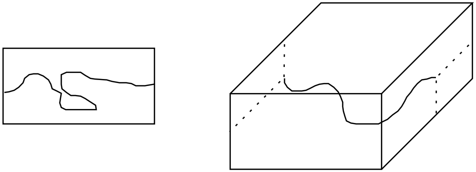
Crossing probabilities

Let Λ be a box in $\mathbb{Z}^d$, with the side lengths not necessarily equal. A crossing is an open path from one side of the box to the other.

Easy way



Hard way



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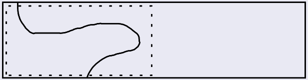

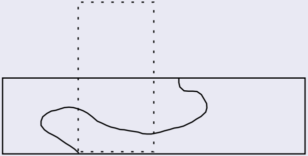

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By the previous theorem we know that the box $[-n/2, n/2] \times [-n, n] \times \cdots \times [-n, n]$ has an easy-way crossing with probability at least c.

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In $d = 2$ Kesten improved this to $n^{-1/3}$.

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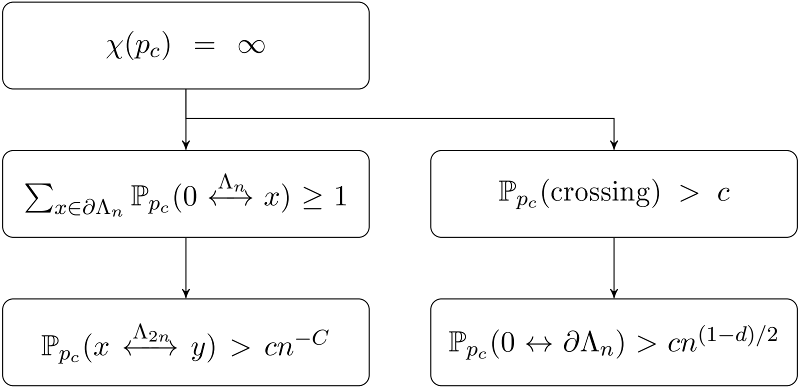
Dependencies diagram



The Aizenman-Kesten-Newman argument

Exploration and martingales

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Let E be the number of open edges in $\mathscr{C}(0)$ and let B be the number of closed edges in its boundary. Let $\lambda > 0$ be some parameter. Then

$$\mathbb{P}_p(B + E \leq n, |(1-p)E - pB| > \lambda\sqrt{n}) \leq Ce^{-c\lambda^2}.$$

Exploration and martingales

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Let E be the number of open edges in $\mathscr{C}(0)$ and let B be the number of closed edges in its boundary. Let $\lambda > 0$ be some parameter. Then

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We define sets of edges $\emptyset = S_0 \subset S_1 \subset \cdots$ for $i \leq n$ as follows.

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Notation

Let A, B be subsets of $E \subseteq \mathbb{Z}^d$. We denote by

$$A \overset{E}{\Longleftrightarrow} B$$

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Theorem

Let V be the number of edges (x, y) in Λ_n such that $\{x, y\} \Leftrightarrow \partial\Lambda_n$ i.e. both x and y are connected to $\partial\Lambda_n$ but $x \overset{\Lambda_n}{\not\leftrightarrow} y$.

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"with high probability" can be made to mean "with probability $> 1 - n^{-1/2}$" and we are done. $\square$

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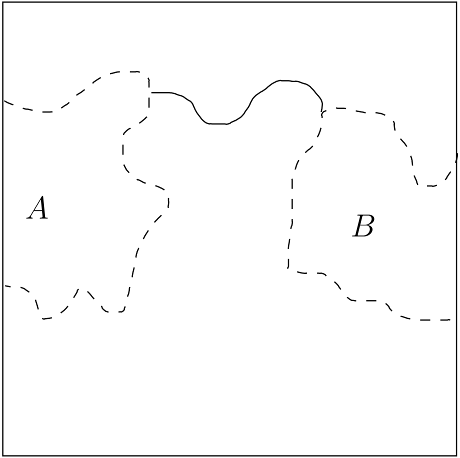

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Proof.

Let $x \in A \cap \Lambda_k$ and $y \in B \cap \Lambda_k$. With probability at least ck^{2d-2d^2} there is an open path γ from x to y. The portion of γ from its last vertex in A until the first vertex in B after it demonstrates the lemma. □

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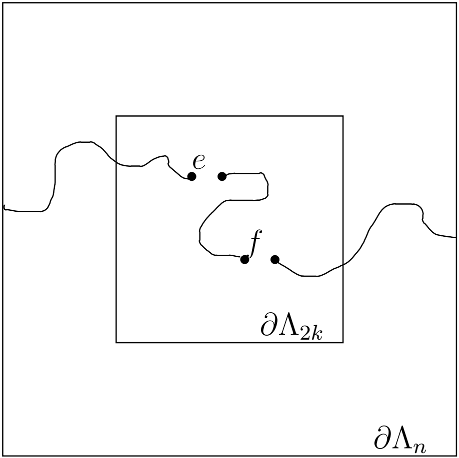

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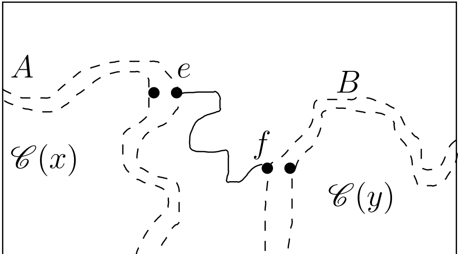

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This kind of argument is called a "patching argument".

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Choosing $k = n^{1/(8d^2+8d)}$ proves the theorem. □

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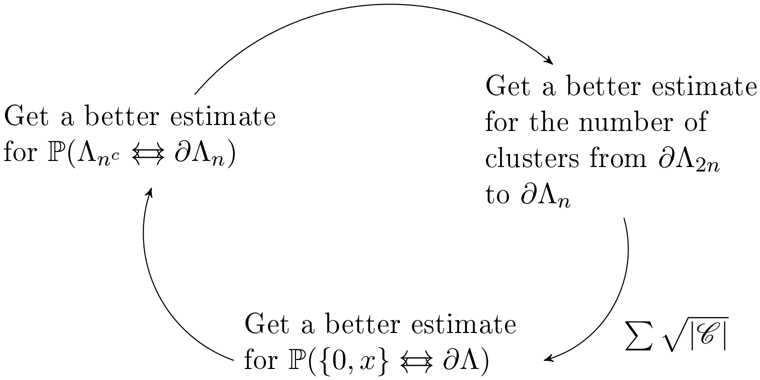

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Definition

Let η be some positive number smaller than $\frac{1}{8d^2+8d}$.

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By continuity, the same inequality will hold for a slightly smaller p. By a theorem of Liggett, Schonmann and Stacey (1997), if c_1 is sufficiently small and n sufficiently large, then an infinite cluster exists, contradicting $p < p_c$. □

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By the lemma,

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$$\#\{\text{such boxes}\} \geq cn^{-d\nu} \sum_{\mathscr{C}\text{ small}} N(\mathscr{C})^{(d-1)/d}.$$

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Together these give

$$\mathbb{E} \sum_{\mathscr{C} \text{ small}} N(\mathscr{C})^{(d-1)/d} \leq Cn^{d/2+C\nu} + Cn^{C\nu} \sum_{\mathscr{C}} \mathbb{E}\sqrt{N(\mathscr{C})}.$$

Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_{n^\nu} \nleftrightarrow \partial\Lambda_n) > Cn^{-d}$.

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Let $N(\mathscr{C})$ be the number of n^ν-subboxes of $\Lambda_{n/2}$ that intersect $\mathscr{C}$. $\mathbb{E}\sum_{\mathscr{C} \text{ small}} N(\mathscr{C})^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}$

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For ν sufficiently small, we reach a contradiction. □

Theorem (Duminil-Copin-K-Tassion, unpublished)

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The proof in a nutshell

The Aizenman-Kesten-Newman-Cerf argument gives

$$\mathbb{P}(\Lambda_{n^\nu} \Leftrightarrow \Lambda_n) \leq \text{uninteresting terms } n^{-d} \sum \sqrt{|\mathscr{C}|}.$$

The contradictory assumption, the isoperimetric inequality and the fact that there are no large clusters give

$$\mathbb{P}(\Lambda_{n^\nu} \Leftrightarrow \Lambda_n) \geq \text{uninteresting terms } n^{-d} \sum |\mathscr{C}|^{(d-1)/d}.$$

And these two contradict.

Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_{n^\nu} \nleftrightarrow \partial\Lambda_n) > Cn^{-d}$.

- Going through the calculation gives

$$\nu < \frac{d-2}{d^3 + 4d^2 + d - 2}$$

so, say, $1/64$ at $d = 3$.

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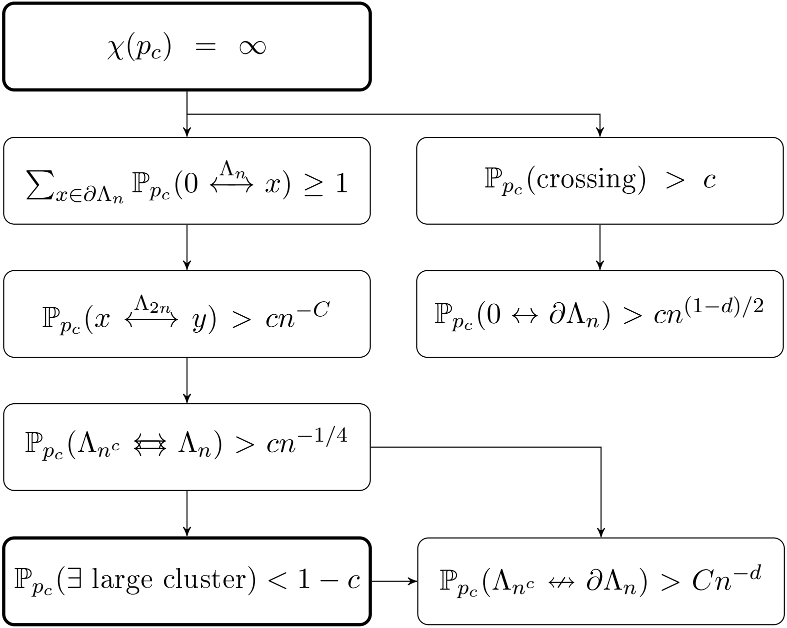
- The theorem holds also at $d = 2$ (known since the 80s, with a different proof).

Dependencies diagram II



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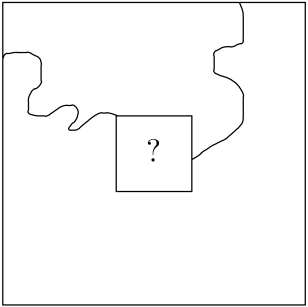

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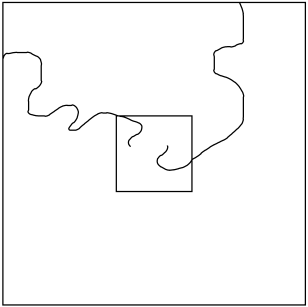

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Let η be sufficiently small so that $\mathbb{P}(\Lambda_{n^\eta} \Leftrightarrow \Lambda_n) \leq Cn^{-1/4}$. Let γ be sufficiently small so that $\mathbb{P}(\Lambda_{n^\gamma} \leftrightarrow \partial\Lambda_{n^\eta}) > cn^{-1/8}$.

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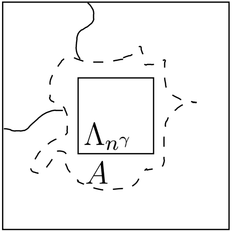

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$\mathbb{P}(B = \mathscr{C}(\Lambda_{n^\gamma}), A \xleftrightarrow{\Lambda_n \setminus A} \partial \Lambda_n) \geq P \cdot \mathbb{P}(B = \mathscr{C}(\Lambda_{n^\gamma}))$.

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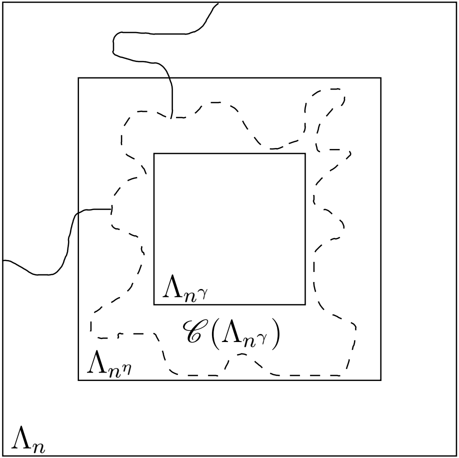
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or $P < Cn^{-1/8}$. □

Theorem (Chayes, Chayes, Newman, Grimmett, Kesten, Schonmann...)

For $p < p_c$ there is a number, denoted by $\xi(p)$, such that

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We will only show a lemma from proof, to demonstrate yet another use of Cerf's theorem.

Lemma

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Proof.

Let m be such that $(1-\theta)^m < \frac{1}{3}\varepsilon$.

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Let m be such that $(1-\theta)^m < \frac{1}{3}\varepsilon$. Let k be so large such that

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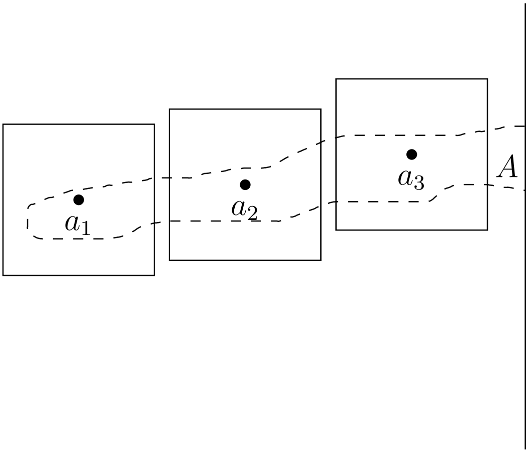

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On the other hand

$$\mathbb{P}(\forall i : a_i + \Lambda_k \leftrightarrow \infty, a_i + \Lambda_k \not\leftrightarrow a_i + \Lambda_K) > 1 - \frac{2\varepsilon}{3}.$$

□

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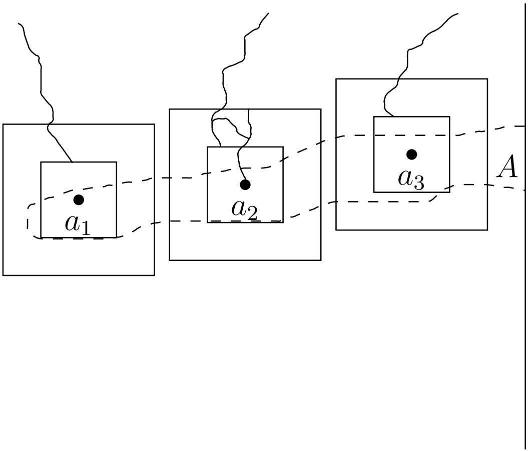


Thanks for your attention!