OOPS 3

Tuesday, August 3, 2021 8:15 PM

I. Review

Showed

$$\mathrm{Var}(\log Z^{\theta}_{N,t}) \leq C N^{2/3}$$

using i) existence of invariant measure (O'Connell-Yor)

ii) integration by parts

$$\mathbb{E}[\log Z^{\theta}_{N,t} B_0] = \mathbb{E}[s_0^+]$$

iii) estimate $\mathbb{E}[s_0^+]$ by varying parameter θ

II. Alternative approach to OY polymer.

$$Z^{\theta}_{N,t} = \int_{-\infty < s_0 < \cdots < s_{N-1} < t} e^{s_0\theta - B_0(s_0) + \cdots + B_N(t) - B_N(s_{N-1})} \, d\underline{s}$$

Itô's formula:

$$dZ^{\theta}_{N,t} = Z^{\theta}_{N-1,t}\, dt + Z^{\theta}_{N,t}\, dB_N + \frac{1}{2} Z^{\theta}_{N,t}\, dt$$

$$d \log Z^{\theta}_{N,t} = e^{-\log Z_{N-1} + \log Z_N} dt + dB_N$$

Define: $r_j = \log Z_j - \log Z_{j-1} \quad j = 1, \ldots, N$

(Recall $Z_0 := e^{B_0 - \theta t}$.)

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$$\Rightarrow \begin{cases} dr_1 = (e^{-r_1} - \theta)\,dt + dB_0 + dB_1 \\ dr_j = (e^{-r_j} - e^{-r_{j-1}})\,dt + dB_j - dB_{j-1}, \quad j \geq 2. \end{cases}$$

This system (or similar ones) have appeared in several places in the literature: Ferrari, Spohn, Weiss, O'Connell, Diehl, Gubinelli, Jara, Moreno-Flores, ...

If we let $V(x) = e^{-x}$, we have:

$$\begin{cases} dr_1 = (-V'(r_1) - \theta)\,dt + dB_0 + dB_1 \\ dr_j = (-V'(r_j) + V'(r_{j-1}))\,dt + dB_j - dB_{j-1}. \end{cases}$$

Assuming the system of SDEs has a solution:

Generator

$$L = \frac{1}{2}\partial_{r_1}^2 + \frac{1}{2}\sum_{j=2}^{N} (\partial_{r_j} - \partial_{r_{j-1}})^2 + \frac{1}{2}\partial_{r_N}^2$$

$$- (V'(r_1) + \theta)\partial_{r_1} + \sum_{j=2}^{N} (V'(r_{j-1}) - V'(r_j))\partial_{r_j}.$$

Define

product form

$$\omega = \prod_{j=1}^{N} \frac{e^{-\theta r_j}\, e^{-V(r_j)}}{Z(\theta)},$$

$$Z(\theta) = \int e^{-\theta x} e^{-V(x)} dx \quad \left(\begin{array}{l} = \Gamma(\theta) \text{ in OY case} \\ V(x) = e^{-x} \end{array} \right)$$

By direct computation,

$$\mathcal{L}^* \omega = 0$$

where $\mathcal{L}^*$ is the formal adjoint of $\mathcal{L}$, so ω is an invariant measure.

$$\mathbb{E}^{\omega}[F(r_1(0), \ldots, r_N(0))] = \mathbb{E}^{\omega}[F(r_1(t), \ldots, r_N(t))].$$

In case $V(x) = e^{-x}$, we recover stationarity of the increments $\log Z_j - \log Z_{j-1}$.

III. Height function

Analysis in previous lectures was based on decomposition:

$$\log Z_{N,t}^{\theta} = \sum_{j=1}^{N} \underbrace{r_j(t)}_{= \log Z_j - \log Z_{j-1}} - B_0(t) + \theta t$$

For general V, we can also consider:

correlated

$$W_{N,t}^{\theta} := \sum_{j=1}^{N} \underbrace{r_j(t)}_{\text{solutions to SDE}} - B_0(t) + \theta t$$

It is expected that for "most" V, $W_{N,t}^{\theta}$ has KPZ

It is expected that for "most" V, $W_{N,t}^{\theta}$ has KPZ fluctuations (e.g. Ferrari-Spohn-Weiss, Sasamoto-Spohn ...).

Try to follow the proof for polymers.

$$\mathrm{Var}(W_{N,t}^{\theta}) \underset{\substack{\uparrow \\ \text{arithmetic}}}{=} \mathrm{Var}\Big(\sum_{j=1}^{N} r_j\Big) - t + 2\,\mathbb{E}\big[W_{N,t}^{\theta} B_0(t)\big]$$

How to analyze $\mathbb{E}[W_{N,t}^{\theta} B_0(t)]$?

For polymer, we used Gaussian integration by parts:

$$\mathbb{E}[\log Z \, B_0(t)] = \frac{d}{d\delta}\mathbb{E}\Big[\log \int_{-\infty<s_0<\cdots<t} e^{\theta s_0 - \delta s_0^{+} - B_0(s_0) + \cdots} \, d\underline{s}\Big]$$

$$= \mathbb{E}\big[E[s_0^{+}]\big]$$

If we start from the equations, not clear what $E[\cdot]$ would be, but we can still integrate by parts.

$$\mathbb{E}[W_{N,t}^{\theta} B_0(t)] = \frac{d}{d\delta}\mathbb{E}\Big[W_{N,t}^{\theta}\big[B_0(s) + \delta s^{+}, 0<s<t\big]\Big]\Big|_{\delta=0}$$

Formally:

$$\mathrm{Var}(W_{N,t}^{\theta}) = N\psi_1(\theta) - t + 2\,\mathbb{E}\big[\partial_{\theta,eq} W_{N,t}^{\theta}\big]$$

$$\frac{d^2}{d\theta^2} \log Z(\theta)$$

$\partial_{\theta, eq} W^{\theta}_{N,t}$: differentiate with respect to parameter θ only for $t \geq 0$ (not initial data).

IV. Computing $\mathbb{E}[\partial_{\theta, eq} W^{\theta}_{N,t}]$.

In case $V(x) = e^{-x}$, we can compute $W^{\theta}_{N,t}$ explicitly and

$$\partial_{\theta, eq} \log Z^{\theta}_{N,t}$$

$$= \partial_\theta \log \int_{-\infty < s_0 < \cdots < t} e^{-\eta s_0^- + \theta s_0^+ - B_0(s_0) + \cdots + B_N(t) - B_N(s_{N-1})} ds \Big|_{\eta = \theta}.$$

In general, not possible to compute exactly, but we can differentiate the equations:

$$\begin{cases} dr_1 = (-V'(r_1) - \theta) dt + dB_0 + dB_1 \\ dr_j = (-V'(r_j) + V'(r_{j-1})) dt + dB_j - dB_{j-1} \\ r_j(0) \sim \text{stationary} \end{cases}$$

$$\Big\downarrow \partial_\theta$$

$$\begin{cases} d v_1 = (-V''(r_1) v_1 - 1) dt \\ d v_j = (-V''(r_j) v_j + V''(r_{j-1}) v_{j-1}) dt \\ v_j(0) = 0 \end{cases}$$

The equations are linear but depend on solutions:

$$\Rightarrow \quad v_1(t) = -\int_0^t e^{-\int_s^t V''(r_1(u))\,du}\, ds \leq 0$$

$$v_j(t) = \int_0^t e^{-\int_s^t V''(r_j(u))\,du}\, V''(r_{j-1}(s))\, v_{j-1}(s)\, ds \leq 0$$

if $V'' \geq 0$.

If $V(x) = \frac{x^2}{2} \Rightarrow V'' = 1$, we can compute exactly:

$$\mathbb{E}[\partial_{\theta,eq} W^\theta_{N,t}] = \int_0^t s^{N-1} e^{-s} \frac{ds}{(N-1)!}$$

$$\mathrm{Var}(W^\theta_{N,t}) \sim N^{1/2} \text{ if } t = N.$$

not $2/3$.

In general, we use monotonicity properties:

$$d W^\theta_{N,t} = -V'(r_N)\,dt + dB_N$$

$$\Big\downarrow \partial_\theta$$

$$d\, \partial_\theta W^\theta_{N,t} = -V''(r_N)\, \underbrace{v_N}_{\leq 0}\, dt \geq 0$$

So parameter in initial data

parameter in SDE

$\theta \mapsto W^{\eta,\theta}_{N,t}$ is increasing.

Can also show: if $V'' \geq 0$, $V''' \leq 0$, then

$$\theta \mapsto W_{N,t}^{\eta,\theta} \quad \text{convex.}$$

Want to apply the strategy we used for OY polymer:

$$\begin{aligned}
\mathrm{Var}(\log Z_{N,t}^{\theta}) &= 2\,\mathbb{E}[\,E[s_0^+]\,] \\
&\lesssim \mathbb{E}[\,E[s_0]^2]^{1/2} + O(N^{1/2}) \\
&\leq \frac{1}{h}\,\mathbb{E}[\,(\log Z^{\theta+h} - \log Z^{\theta})^2\,]^{1/2} + O(N^{1/2}) \\
&\quad + \frac{1}{h}\,\mathbb{E}[\,(\log Z^{\theta-h} - \log Z^{\theta})^2]^{1/2}
\end{aligned}$$

Concexity

$\underline{V(x)=e^{-x}\ (\mathrm{OY})}$	$\underline{\text{General } V}$
$\log Z_{N,t}^{\theta}$	$W_{N,t}^{\theta}$
$E[s_0^+]$	$\partial_{\theta,eq} W_{N,t}^{\theta}$
$E[s_0]$	$\frac{d}{d\theta} W_{N,t}^{\theta}$

V. Initial data

For polymer, initial data is built in:

$$Z_{N,t}^{\eta,\theta} = \int e^{\theta s_0^+ - \eta s_0^- - B_0(s_0) + \cdots + B_N(t) - B_N(s_{N-1})}\, ds$$

θ

$$Z^{\eta,\theta}_{N,t} = \int_{-\infty < s_0 < \cdots < t} e^{\theta s_0^+ - \eta s_0^- - B_0(s_0) + \cdots + B_N(t) - B_N(s_{N-1})} \, d\underline{s}$$ θ

For general V, we define

$$W_{N,t} = \sum_{j=1}^{N} r_j(t) - B_0(t) + \theta t$$

where
$$\begin{cases} dr_1 = -V'(r_1)\,dt - \theta\,dt + dB_0 + dB_1 \\ dr_j = (-V'(r_j) + V'(r_{j-1}))\,dt + dB_j - dB_{j-1} \\ r_j(0) \sim f_j(\eta) \end{cases}$$

Want to choose initial data so $\eta \mapsto W^{\eta,\theta}_{N,t}$ has good properties. For example:

$$\frac{d^2}{d\theta^2} W^{\theta,\theta}_{N,t} \geq 0.$$

<u>Idea:</u> use the equations to create initial data:

$$f_j(\eta) = \lim_{T\to\infty} \tilde{r}_j^{\eta}(T)$$

$$\begin{cases} d\tilde{r}_1 = (-V'(\tilde{r}_1) - \eta)\,dt + dB_0 + dB_1 \\ d\tilde{r}_j = (-V'(\tilde{r}_j) + V'(\tilde{r}_j))\,dt + dB_j - dB_{j-1} \end{cases}$$

<u>Claim</u>: with this choice,

$$\partial_\eta W^{\eta,\theta}_{N,t} \leq 0,$$

$$\partial_\eta^2 W^{\eta,\theta}_{N,t} \geq 0,$$

$$\partial_\eta \partial_\theta W_{N,t}^{\eta,\theta} \geq 0.$$

VI. Estimate

$$\mathrm{Var}(W_{N,t}^{\theta,\theta}) = N\psi_1(\theta) - t + 2\mathbb{E}[\partial_{\theta,eq} W_{N,t}^{\theta,\theta}]$$

$$\leq N\psi_1(\theta) - t + 2\mathbb{E}\left[\left(\frac{d}{d\theta} W_{N,t}^{\theta,\theta}\right)^2\right]^{1/2} + O(N^{1/2})$$

$$\leq N\psi_1(\theta) - t$$

$$+ \frac{1}{h}\mathbb{E}\left[\left(W_{N,t}^{\theta\pm h} - W_{N,t}^{\theta}\right)^2\right]^{1/2} + O(N^{1/2})$$

If: $\circ \; \left|\mathbb{E}[W_{N,t}^{\theta+h}] - \mathbb{E}[W_{N,t}^{\theta}]\right| \leq CNh^2$ ✓ (same as OY)

$\circ \; \left|\mathrm{Var}(W_{N,t}^{\theta+h}) - \mathrm{Var}(W_{N,t}^{\theta})\right| \leq CNh$ (use coupling)

$$\mathrm{Var}\left(W_{N,t}^{\theta+h}\right) = N\psi_1(\theta) - t + 2\mathbb{E}[\partial_{\theta,eq} W_{N,t}]$$

increasing in parameter in equations

decreasing in parameter in initial data

$$\Rightarrow \quad \mathrm{Var}(W_{N,t}^{\theta}) \leq N^{2/3} + N^{1/3}\left(\mathrm{Var}(W_{N,t}^{\theta})\right)^{1/2}$$

if $|N\psi_1(\theta) - t| \leq CN^{2/3}$.

$$\Rightarrow \quad \mathrm{Var}(W_{N,t}^{\theta}) \leq CN^{2/3}.$$

$$\Rightarrow \quad \mathrm{Var}(W_{N,t}^{\theta}) \leq C N^{2/3}.$$

Remains to show:

$$\mathbb{E}[\partial_{\theta,eq} W_{N,t}^{\theta,\theta}] \leq \mathbb{E}\left[\left(\frac{d}{d\theta} W_{N,t}^{\theta,\theta}\right)^2\right]^{1/2} + O(N^{1/2})$$

O.Y.

$$\mathbb{E}[E[s_0^+]] \leq \mathbb{E}[E[s_0]^2]^{1/2} + O(N^{1/2})$$

Key estimate:

$$E[s_0^+]E[s_0^-] \leq C\, E[(s_0 - E[s_0])^2]$$

$$-\partial_\theta W_{N,t}^{\eta,\theta}\, \partial_\eta W_{N,t}^{\eta,\theta}\Big|_{\eta=\theta} \leq C \frac{d^2}{d\theta^2} W_{N,t}^{\theta,\theta}.$$

With Noack and Landon, we show:

$$-c_0 \partial_\theta W_{N,t} \partial_\eta W_{N,t} \leq \partial_{\eta\theta}^2 W_{N,t} \quad \text{if} \quad e^{c_0 x} V''(x) \text{ is non-increasing}$$

Idea of proof:

$$A_n = c_0 \partial_\theta W \partial_\eta W + \partial_{\eta\theta}^2 W \Big|_{\eta=\theta}$$

$$d A_n = -V''(r_n) A_n + V''(r_n) A_{n-1}$$

$$- \left[c_0 V''(r_n) + V'''(r_n) \right] \partial_\xi r_n \partial_\eta r_n$$

$$\Rightarrow \quad d A_n \approx -V''(r_n) A_n$$