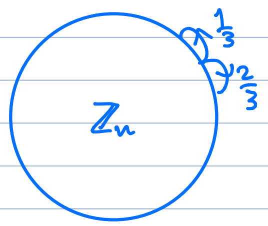
Recall **Thm** (Oliveira, Peres - S.)

$\forall\ \alpha < \frac{1}{2}\ \ \exists\ c_\alpha$ and $c'_\alpha > 0$ s.t. $\forall$ finite reversible lazy MC's

$$c'_\alpha \cdot t_H(\alpha) \leq t_{mix} \leq c_\alpha \cdot t_H(\alpha)$$

Recall $t_H(\alpha) = \max\limits_{x, A : \pi(A) \geq \alpha} \mathbb{E}_x[\tau_A]$

Removing the reversibility assumption



$t_{mix} \asymp n^2$, $t_H(\alpha) \asymp n\ \ \forall \alpha$.

$\max\limits_{x,y} \mathbb{E}_x[\tau_y] \asymp n$

For $\alpha \in (0,1)$ define $\mathcal{A}(\alpha) = \{ A = (A_t)_{t \geq 0} : \pi(A_t) \geq \alpha,\ \forall t \geq 0 \}$

and define also $\tau_A = \inf\{t \geq 0 : X_t \in A_t\}$ for $A = (A_t)_{t \geq 0}$.

Theorem (Winkler - S.)

Fix $\alpha < \frac{1}{2}$. Then there exist two positive constants c_α and c'_α s.t.

for all irreducible finite Markov chains if

$$t_{mov}(\alpha) = \sup_{x, A \in \mathcal{A}(\alpha)} \mathbb{E}_x[\tau_A], \text{ then}$$

$$c_\alpha\, t_{mov}(\alpha) \leq t_{mix} \leq c'_\alpha\, t_{mov}(\alpha).$$

Hitting times (Griffiths, Kang, Oliveira, Patel)

Theorem 1 Let $0 < \alpha < \beta \leq \frac{1}{2}$. Then for every irreducible finite Markov chain

$$t_H(\alpha) \leq t_H(\beta) + \left(\frac{1}{\alpha} - 1\right) t_H(1-\beta) \leq \frac{1}{\alpha} t_H(\beta).$$

(Recall $t_H(\alpha) = \max\limits_{x, A : \pi(A) \geq \alpha} \mathbb{E}_x[\tau_A]$)

Remark These 2 ineq. are sharp, i.e. $\forall\ 0 < \alpha < \beta \leq \frac{1}{2}$ there exists an irred. finite Markov chain for which all three terms are equal.

$\beta = \frac{1}{2}$ is a boundary case, in the sense that $\forall\ \beta > \frac{1}{2}$ there exists a class of irred. finite MC's s.t. $t_H(\alpha)/t_H(\beta)$ can be made arbitrarily large.

Lemma 1 For an irred., finite MC for any $\emptyset \neq A, B \subseteq S$ (state space)

define $d^+(A,B) = \max_{x \in A} \mathbb{E}_x[\tau_B]$ and $d^-(A,B) = \min_{x\in A} \mathbb{E}_x[\tau_B]$.

Then
$$\pi(A) \leq \frac{d^+(A,B)}{d^+(A,B) + d^-(B,A)}.$$

Non-rigorous explanation: $\pi(A)\cdot(d^+(A,B) + d^-(B,A)) \leq d^+(A,B)$.

Let $x \in A$ be s.t. $\mathbb{E}_x[\tau_B] = d^+(A,B)$

$$\pi(A)\left(\mathbb{E}_x[\tau_B] + d^-(B,A)\right) \leq d^+(A,B)$$

ergodic theorem $\rightsquigarrow$ by time t the chain visits the set A : $\pi(A)t$ times

Define $\tau_{B,A} = \min\{t \geq \tau_B : X_t \in A\}$

By time $\tau_{B,A}$ "the chain spends time $\pi(A)\,\mathbb{E}_x[\tau_{B,A}]$" in A

Also the time spent $\leq \mathbb{E}_x[\tau_B]$ (after τ_B no more visits to A)

$$\Rightarrow \quad \pi(A)\,\underbrace{\mathbb{E}_x[\tau_{B,A}]}_{\geq \mathbb{E}_x[\tau_B] + d^-(B,A)} \leq \mathbb{E}_x[\tau_B] = d^+(A,B).$$

Proof of Theorem 1 Fix x and a set A with $\pi(A) \geq \alpha$.

Want to show $\mathbb{E}_x[\tau_A] \leq t_H(\beta) + \left(\frac{1}{\alpha} - 1\right) t_H(1-\beta)$.

Want to define a set B with $\pi(B) \geq \beta$ so that we first wait to hit B and then starting from there the time to hit is controlled by the second term.

Define $B = \{y : \mathbb{E}_y[\tau_A] \leq \left(\frac{1}{\alpha} - 1\right) t_H(1-\beta)\}$.

If we show $\pi(B) \geq \beta$, then we are done, because

$$\mathbb{E}_x[\tau_A] \leq \mathbb{E}_x[\tau_B] + \max_{y\in B} \mathbb{E}_y[\tau_A] \leq t_H(\beta) + \left(\frac{1}{\alpha} - 1\right) t_H(1-\beta).$$

Claim $\pi(B) \geq \beta$.

Suppose not, i.e. $\pi(B) < \beta$ and let $C = B^c$. Then $\pi(C) > 1-\beta$.

$$\pi(A) \leq \frac{d^+(A,C)}{d^+(A,C) + d^-(C,A)} \quad \text{using Lemma}$$

$d^+(A,C) \leq t_H(1-\beta)$ and $d^-(C,A) > \left(\frac{1}{\alpha}-1\right) t_H(1-\beta)$

So $\pi(A) < \dfrac{1}{1+\frac{1}{\alpha}-1} = \alpha$ which is a contradiction, because $\pi(A) \geq \alpha$. $\square$

Proof of Lemma 1

Lemma 2 Let X be an irreducible finite Markov chain with values in S. Let μ be a prob. distribution and τ a stopping time s.t.

$$\mathbb{P}_\mu(X_\tau = x) = \mu(x) \quad \forall x.$$

Then $$\mathbb{E}_\mu\left[\sum_{i=0}^{\tau-1} 1(X_i \in A)\right] = \pi(A) \cdot \mathbb{E}_\mu[\tau], \quad \forall A \subseteq S.$$

Proof Exercise

Hint $$\nu(x) = \mathbb{E}_{x_0}\left[\sum_{i=0}^{\tau_{x_0}^+ - 1} 1(X_i = x)\right] \rightsquigarrow \nu = \nu P \rightsquigarrow \pi$$

$\tau_{x_0}^+ = \min\{t \geq 1 : X_t = x_0\}$

Define $\tilde{\nu}(x) = \mathbb{E}_\mu\left[\sum_{i=0}^{\tau-1} 1(X_i = x)\right]$. Show $\tilde{\nu} = \tilde{\nu} P \Rightarrow \tilde{\nu}$ has to be a multiple of π.

Pf of L.1 Define

$$\tau_{B,A} = \min\{t \geq \tau_B : X_t \in A\}.$$

and an auxiliary Markov chain with transition matrix Q

$$Q(x,y) = \mathbb{P}_x(X_{\tau_{B,A}} = y), \quad x, y \in A.$$

This is a finite irred. MC $\Rightarrow$ it possesses an invar. distr. μ.

Let $\nu(y) = \mathbb{P}_\mu(X_{\tau_B} = y)$, $y \in B$. Call $\tau = \tau_{B,A}$

Count the number of visits to A up until $\tau_{B,A}$ starting from μ.

$$\mathbb{E}_\mu\left[\sum_{i=0}^{\tau-1} 1(X_i \in A)\right] \leq \mathbb{E}_\mu[\tau_B] \qquad \star$$

Because μ is invar. for $Q \Rightarrow \mathbb{P}_\mu(X_\tau = x) = \mu(x) \ \forall x$.

So the conditions of Lemma 2 are satisfied

$$\Rightarrow \mathbb{E}_\mu\left[\sum_{i=0}^{\tau-1} 1(X_i \in A)\right] = \pi(A)\,\mathbb{E}_\mu[\tau] = \pi(A)\left(\mathbb{E}_\mu[\tau_B] + \mathbb{E}_\nu[\tau_A]\right)$$

So $$\pi(A)\left(\mathbb{E}_\mu[\tau_B] + \mathbb{E}_\nu[\tau_A]\right) \underset{\star}{\leq} \mathbb{E}_\mu[\tau_B]$$

$$\Rightarrow \pi(A)\,\mathbb{E}_\nu[\tau_A] \leq (1-\pi(A))\,\mathbb{E}_\mu[\tau_B].$$

This now completes the proof, because

$$\pi(A) \leq \frac{d^+(A,B)}{d^+(A,B) + d^-(B,A)} \Rightarrow \pi(A)\, d^-(B,A) \leq (1-\pi(A))\, d^+(A,B)$$

and $\mathbb{E}_\nu[\tau_A] \geq d^-(B,A)$ and $\mathbb{E}_\mu[\tau_B] \leq d^+(A,B)$

(ν is supported on B and μ is supported on A). $\square$