Exercises from Perla Sousi's OOPS-course

June 15–18

See the slides or video for definitions and context.

Lecture 1

Exercise 1. Show that the random walk on the cycle of length n biased to move to the right with probability 2/3 has $t_{mix} \asymp n^2$ but $t_H(\alpha) \asymp n$ for any α.

Exercise 2. Let G be the graph consisting of two cliques of size n, with a single edge connecting them. Prove that the random walk on this graph has $t_{mix} \asymp n^2$ but for $\alpha > 1/2$ has $t_H(\alpha) \asymp n$.

Exercise 3. Show that $d_G(t)$ is decreasing in t.

Exercise 4. Prove that for reversible Markov chains, $t_{stop} \leq 8t_{mix}$. (Hint: use the searation distance to define a stopping time.)

Lecture 2.

Exercise 5. Prove the following lemma: For an irreducible M.C. $X_\cdot$, measure μ and stopping time τ such that for all x,

$$\mathbb{P}_\mu(X_\tau = x) = \mu(x).$$

Then for any set A,

$$\mathbb{E}_\mu\left[\sum_{i=0}^{\tau-1} 1(X_i \in A)\right] = \pi(A)\mathbb{E}_\mu(\tau).$$

(See hint on slides.)

Exercise 6. Check that if the chain is lazy then $\gamma_* = \gamma$. (See spectral methods note for definitions.)

Exercise 7. (Poincare inequality). Let P be a lazy and reversible matrix with respect to the invariant distribution π. Then for all $f : S \to \mathbb{R}$ and all $t \geq 0$

$$Var_\pi(P^t f) \leq e^{-2t/t_{rel}} Var_\pi(f).$$

(Hint: Use the spectral theorem.)

Lecture 3.

Exercise 8. Prove the second part of the Basu-Hermon-Peres theorem:

$$t_{mix}(1-\epsilon) \leq hit_{1-\epsilon}(1-2\epsilon) + 2t_{rel}log(8/\epsilon^3).$$