1 Spectral decomposition and relaxation time

Let X be a reversible Markov chain on the finite state space S with transition matrix P and invariant distribution π. Let $f, g : S \to \mathbb{R}$. Their inner product is defined to be

$$\langle f, g \rangle_\pi = \sum_x f(x)g(x)\pi(x).$$

Theorem 1.1. *Let P be reversible with respect to π. The inner product space $(\mathbb{R}^S, \langle \cdot, \cdot \rangle_\pi)$ has an orthonormal basis of real-valued eigenfunctions $(f_j)_{j \le |S|}$ corresponding to real eigenvalues (λ_j) and the eigenfunction f_1 corresponding to $\lambda_1 = 1$ can be taken to be the constant vector $(1, \ldots, 1)$. Moreover, the transition matrix P^t can be decomposed as*

$$\frac{P^t(x,y)}{\pi(y)} = 1 + \sum_{j=2}^{|E|} f_j(x) f_j(y) \lambda_j^t.$$

Proof. We consider the matrix $A(x,y) = \sqrt{\pi(x)}P(x,y)/\sqrt{\pi(y)}$ which using reversibility of P is easily seen to be symmetric. Therefore, we can apply the spectral theorem for symmetric matrices and get the existence of an orthonormal basis (g_j) corresponding to real eigenvalues. It is easy to check that $\sqrt{\pi}$ is an eigenfunction of A with eigenvalue 1. Let D be the diagonal matrix with elements $(\sqrt{\pi(x)})$. Then $A = DPD^{-1}$ and it is easy to check that $f_j = D^{-1}g_j$ are eigenfunctions of P and $\langle f_j, f_i \rangle_\pi = \mathbf{1}(i = j)$. So we have $P^t f_j = \lambda_j^t f_j$ and hence

$$P^t(x,y) = (P^t \mathbf{1}_y)(x) = \sum_{j=1}^{|S|} \lambda_j^t f_j(x) \langle f_j, \mathbf{1}_y \rangle_\pi = \sum_{j=1}^{|S|} \lambda_j^t f_j(x) f_j(y) \pi(y).$$

Using that $f_1 = 1$ and $\lambda_1 = 1$ gives the desired decomposition. $\square$

Let P be a reversible matrix with respect to π. We order its eigenvalues

$$1 = \lambda_1 > \lambda_2 \ge \ldots \lambda_{|S|} \ge -1.$$

We let $\lambda_* = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } P, \lambda \ne 1\}$ and define $\gamma_* = 1 - \lambda_*$ to be the absolute spectral gap. The spectral gap is defined to be $\gamma = 1 - \lambda_2$.

Exercise 1.2. *Check that if the chain is lazy then $\gamma_* = \gamma$.*

Definition 1.3. The relaxation time for a reversible Markov chain is defined to be

$$t_{\text{rel}} = \frac{1}{\gamma_*}.$$

Let $f : S \to \mathbb{R}$. We write

$$\mathbb{E}_\pi[f] = \sum_x f(x)\pi(x) \quad \text{and} \quad \text{Var}_\pi(f) = \mathbb{E}_\pi\left[(f - \mathbb{E}_\pi[f])^2\right].$$

Exercise 1.4 (Poincaré inequality)**.** *Let P be a lazy and reversible matrix with respect to the invariant distribution π. Then for all $f : S \to \mathbb{R}$ and all $t \ge 0$*

$$\text{Var}_\pi\left(P^t f\right) \le e^{-2t/t_{\text{rel}}} \text{Var}_\pi(f).$$

(Hint: Use the spectral theorem)