# The *-Edge Reinforced random walk, Bayesian statistics and statistical physics

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Open Online Probability School, 12-15 July 2021

# Outline I

# Outline II

- VRJP: three timescales
- Partial exchangeability of VRJP
- Bayesian approach to find limit measure of VRJP
- VRJP $\longleftrightarrow$ random Schrödinger operator
- Applications : recurrence/transience

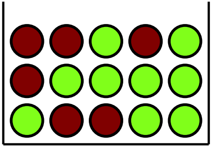
## *-Edge-Reinforced Random Walk (*-ERRW)

- Definition and remarks
- *-ERRW: partial exchangeability
- *-ERRW: statistical view
- *-ERRW: results
- Correspondence *-ERRW $\longleftrightarrow$ *-VRJP
- Limiting manifold and measure
- *-VRJP: Random Schrödinger version

# Pólya urn: definition

- Introduced by Eggenberger and Pólya in 1923: "Über die Statistik verketteter Vorgänge", i.e. "On statistics of linked behaviors".
- Urn with balls of two colors: green and red.
- Initially $a$, resp. $b > 0$ balls of green, red color.
- $G_n$, $R_n$ numbers of balls of green, red color added until $n$-th draw, $G_0 = R_0 = 0$.
- Reinforcement rule: pick one ball at random and put it back together with another ball of same color:

$$\mathbb{P}(G_{n+1} = G_n + 1 \,|\, G_k, R_k \; k \leqslant n) = \frac{a + G_n}{a + G_n + b + R_n} =: \alpha_n.$$



# The notion of exchangeability (de Finetti)

## Definition

Let $(X_i)_{i \geqslant 1}$ random process taking values in $\{0, 1\}$. Then $X$ is called exchangeable if, for all $n \in \mathbb{N}$ and $\sigma \in S_n$,

$$\mathcal{L}\left((X_{\sigma(i)})_{1 \leqslant i \leqslant n}\right) = \mathcal{L}\left((X_i)_{1 \leqslant i \leqslant n}\right).$$

## Theorem (de Finetti)

*If $(X_i)_{i \geqslant 1}$ is exchangeable, then there exists a random variable $\alpha \in [0, 1]$ such that*

$$\frac{1}{n} \sum_{i=1}^{n} X_i \longrightarrow_{n \to \infty} \alpha.$$

*Conditionally on $\alpha$, $(X_i)_{i \geqslant 1}$ is an i.i.d. sequence of Bernoulli random variables with success probability $\alpha$, which we call $P^{\alpha}$.*

# Exchangeability of Pólya urn

Let $\mathbb{P}^{a,b}$ be the law of the Pólya urn starting from $a$, resp. $b > 0$ balls of green, red color.

If $a \in \mathbb{R}$, $n \in \mathbb{N}$, define

$$(a, n) = a \dots (a + n - 1) = \prod_{i=0}^{n-1} (a + i) = \frac{\Gamma(a + n)}{\Gamma(a)},$$

where $\Gamma$ is the Gamma function, and

$$C(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}.$$

## Lemma

*Let $(G_n)_{n\geqslant 0}$ be a Pólya urn, and set $X_n := G_n - G_{n-1}$. Then $(X_n)_{n\geqslant 1}$ is exchangeable and, if $p = \sum_{i=1}^{n} \epsilon_i$, $q = n - p$, then*

$$\mathbb{P}^{a,b}(X_i = \epsilon_i,\ 1 \leqslant i \leqslant n) = \frac{C(a + p, b + q)}{C(a, b)}.$$

# Pólya urn: statistical view

- Given sequence of i.i.d. Bernoulli random variables with unknown random success probability $\alpha$, how can we estimate $\alpha$?
- Bayesian approach: choose prior distribution on random variable $\alpha$.
- Let $\mathcal{L}(a, b)$ be the law of the random variable $\alpha$ under $\mathbb{P}^{a,b}$.
- If prior on $\alpha$ is $\mathcal{L}(a, b)$, then by definition

$$\mathcal{L}^{\alpha \sim \mathcal{L}(a,b)}(P^{\alpha}((\mathbf{1}_{\text{success at time } n})_{n \geqslant 1})) = \mathbb{P}^{a,b}((G_n - G_{n-1})_{n \geqslant 1}),$$

where $(G_n)_{n \in \mathbb{N}}$ defined from Pólya urn above.

# Statistical view of Pólya urn: consequences

- Hence, if the prior on $\alpha$ is $\mathcal{L}(a, b)$, then the posterior distribution conditioned on $p$ successes and $q$ failures is $\mathcal{L}(a+p, b+q)$, as the distribution of $\alpha$ for the Pólya urn starting from $a+p$ green balls and $b+q$ red balls.
- The prior and posterior are in the same family of probability distributions, and are thus called conjuguate priors.
- $(G_n, R_n)$ is a sufficient statistic for $\alpha$ at time $n$:
  - Informally: no other statistic that can be calculated from the sequence $(G_k)_{k \leqslant n}$ provides any additional information as to the value of the parameter $\alpha$.
  - Formally: given statistical model $\{P_\alpha : \alpha \in (0,1)\}$, where $P_\alpha$ is the law of i.i.d. sequences with success probability $\alpha$, $P_\alpha((G_k)_{k \leqslant n} | G_n)$ does not depend on $\alpha$ (Exercise).
  - It is a minimal sufficient statistics: there is no sufficient statistics that needs less information.

# Pólya urn: how to find $\mathcal{L}(a,b)$? (I)

- Assume that the law of $\alpha$ under $\mathbb{P}^{a,b}$ has a smooth integrable density measure $\varphi^{a,b}$ w.r.t. Lebesgue measure on $[0,1]$.
- Under $\mathbb{P}^{a,b}$, the posterior distribution conditioned on $p$ successes and $q$ failures is $\mathcal{L}(a+p,b+q)$.
- But, if $p=\sum_{i=1}^n \epsilon_i$, $q=n-p$, then

$$\mathbb{P}^{a,b}\left(\alpha\in[x,x+dx]\,|\,X=\epsilon|_{1\leqslant i\leqslant n}\right)=\frac{\varphi^{a,b}(x)x^p(1-x)^q}{\mathbb{P}^{a,b}\left(X_i=\epsilon_i,\;1\leqslant i\leqslant n\right)}\,dx$$

$$=\frac{\varphi^{a,b}(x)x^p(1-x)^q C(a,b)}{C(a+p,b+q)}dx.$$

- It follows that

$$\varphi^{a+p,b+q}(x)=\frac{x^p(1-x)^q C(a,b)}{C(a+p,b+q)}\varphi^{a,b}(x)$$

$$=\left(\frac{x^{a+p}(1-x)^{b+q}}{C(a+p,b+q)}\right)\left(\frac{x^a(1-x)^b}{C(a,b)}\right)^{-1}\varphi^{a,b}(x).$$

# Pólya urn: how to find $\mathcal{L}(a, b)$? (II)

Let

$$n = p + q, \ \ \beta(n) = \frac{a+p}{a+b+n}.$$

Then, using Stirling's approximation

$$\Gamma(z) \sim_{z\to\infty} \sqrt{2\pi} z^{z-1/2} e^{-z},$$

$$\begin{aligned} C(a+p, b+q) &\sim_{p,q\to\infty} \sqrt{2\pi} \frac{(a+p)^{a+p-1/2}(b+q)^{b+q-1/2}}{(a+b+n)^{a+b+n-1/2}} \\ &\sim_{p,q\to\infty} \sqrt{2\pi}\, (n\beta(n)(1-\beta(n)))^{-1/2}\, \eta_{\beta(n)}(\beta(n))^n, \end{aligned}$$

where

$$\eta_\beta(x) = x^\beta (1-x)^{1-\beta}.$$

# Pólya urn: how to find $\mathcal{L}(a, b)$? (III)

Therefore

$$\varphi^{a+p,b+q}(x) \underset{p,q\to\infty}{\sim} \frac{\varphi^{a,b}(x)C(a,b)}{x^a(1-x)^b}\frac{1}{\sqrt{2\pi}}\left(n\beta(n)(1-\beta(n))\right)^{1/2}\left(\frac{\eta_{\beta(n)}(x)}{\eta_{\beta(n)}(\beta(n))}\right)^n.$$

Now, for all $\beta \in (0,1)$,

$$\begin{aligned}\log\left(\frac{\eta_\beta(x)}{\eta_\beta(\beta)}\right) &= \beta\log\left(\frac{x}{\beta}\right) + (1-\beta)\log\left(\frac{x}{1-\beta}\right)\\ &= -\frac{(x-\beta)^2}{2\beta(1-\beta)} + o\left((x-\beta)^3\right)\end{aligned}$$

# Pólya urn: how to find $\mathcal{L}(a,b)$? (IV)

We conclude that

$$\lim_{n\to\infty,\,\beta(n)\to x} \int_0^1 \varphi^{a+p,b+q}(y)\,dy = 1 = \frac{\varphi^{a,b}(x)C(a,b)}{x^{a-1}(1-x)^{b-1}},$$

using that

$$\frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{\beta(n)(1-\beta(n))}}\int_{-\infty}^{\infty} \exp\left(-\frac{nz^2}{2\beta(n)(1-\beta(n))}\right)\,dz = 1$$

Therefore

$$\varphi^{a,b}(x) = \frac{x^{a-1}(1-x)^{b-1}}{C(a,b)}.$$

# Edge-Reinforced Random Walk (Coppersmith and Diaconis, 1986)

- $G = (V, E)$ non-oriented locally finite graph
- $a_e > 0$, $e \in E$, initial weights

• Edge-Reinforced Random Walk (ERRW) $(X_n)$ on $V$ : $X_0 = i_0$ and, if $X_n = i$, then

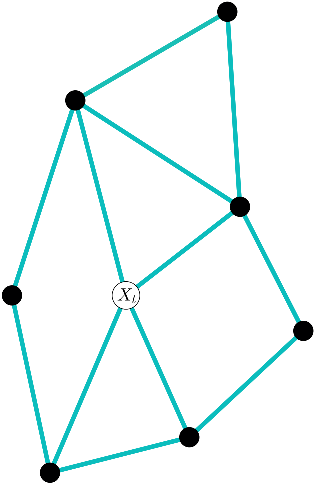
$$\mathbb{P}(X_{n+1} = j \,|\, X_k,\, k \leqslant n) = \mathbb{1}_{\{j\sim i\}} \frac{Z_{\{i,j\}}(n)}{\sum_{k\sim X_n} Z_{\{ik\}}(n)}$$

where

$$Z_{\{i,j\}}(n) = a_{i,j} + \sum_{k=1}^{n} \mathbb{1}_{\{X_{k-1},X_k\}=\{i,j\}}.$$

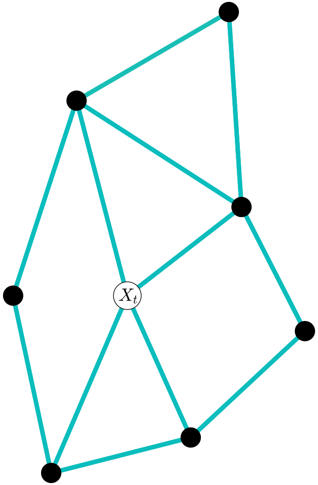
- $a_e$ small: strong reinforcement
- $a_e$ large: small reinforcement

# The ERRW




A simulation due to Andrew Swan.

# The Mixing Measure of ERRW




A simulation due to Andrew Swan.

# The notion of partial exchangeability (Diaconis and Freedman, 1980) (I)

**Definition**

Let $(Y_n)_{n\geqslant 0}$ a random process on a graph $G=(V,E)$. It is called partially exchangeable (resp. reversibly partially exchangeable) if, for any nearest-neighbor path $\gamma=(\gamma_0,\dots,\gamma_n)$ on $V$,

$$\mathbb{P}[(Y_0,\dots,Y_n)=(\gamma_0,\dots,\gamma_n)]$$

only depends on its starting point and on the number of crossings of directed (resp. undirected) edges by $\gamma$.

# The notion of partial exchangeability (Diaconis and Freedman, 1980) (II)

**Theorem (Diaconis and Freedman, 1980)**

*If $(Y_n)_{n\geqslant 0}$ is a.s. recurrent (i.e. $Y_n = Y_0$ infinitely often) and partially exchangeable (resp. reversibly partially exchangeable, and each edge is traversed is traversed in both directions with probability 1) then it is a mixture of Markov chains (resp. reversible Markov chains), i.e.*

$$\mathcal{L}(Y) = \int P^{\omega}(.)\, d\mu(\omega),$$

*Here $P^{\omega}$ denotes the Markov Chain with transition probability $\omega(i,j)$ from $i$ to $j$. If $P^{\omega}$ is reversible, then there exists $x = (x_e) \in (0,\infty)^E$ such that*

$$\omega(i,j) = \omega^x(i,j) = \frac{x_{ij}}{x_i},\ x_i = \sum_{j\sim i} x_{ij}. \quad \textit{Let } P^x = P^{\omega^x}.$$

# Edge-Reinforced random walk (ERRW): partial exchangeability

Let $\mathbb{P}^{a,i_0}$ be the law of ERRW with initial weights $a = (a_e)_{e \in E}$ and starting from $i_0$, also denoted by $ERRW(i_0, a)$.

### Lemma

*The ERRW is reversibly partially exchangeable: more precisely,*

$$\begin{aligned}
&\mathbb{P}^{i_0,a}(X_0 = i_0, \ldots, X_n = i_n = j_0) \\
&= \frac{\prod_{e \in E} (a_e, n_e)}{\prod_{i \in V} 2^{v_i - \delta_{j_0}(i)} \left( \frac{a_i + 1 - \delta_{i_0}(i)}{2}, v_i - \delta_{j_0}(i) \right)} = \frac{\gamma(i_0, a)}{\gamma(j_0, \alpha)},
\end{aligned}$$

*where $n_e$ (resp. $v_i$) is the number of crossings (resp. visits) of edge $e$ (resp. site $i$) by the path $(i_0, \ldots, i_n)$, $\alpha = (a_e + n_e)_{e \in E}$, and*

$$\gamma(i_0, a) = \frac{\prod_{i \in V} \Gamma\left(\frac{1}{2}(a_i + 1 - \delta_{i_0}(i))\right) 2^{\frac{1}{2}(a_i - \delta_{i_0}(i))}}{\prod_{e \in E} \Gamma(a_e)}.$$

# Edge-Reinforced random walk (ERRW): partial exchangeability

In the last equality we use that

$$n_i := \sum_{j \sim i} n_{ij} = 2v_i - \delta_{i_0}(i) - \delta_{j_0}(i)$$

so that

$$v_i - \delta_{j_0}(i) = \frac{(a_i + n_i) - \delta_{j_0}(i)}{2} - \frac{a_i - \delta_{i_0}(i)}{2}.$$

By the Theorem of Diaconis and Freedman (1980), since $ERRW(i_0, a)$ is reversibly partially exchangeable it is a mixture of reversible Markov chains $P^x$.
Let $\mathcal{L}(i_0, a)$ be the mixing measure of $x$ under $\mathbb{P}^{i_0,a}$.

# Edge-Reinforced random walk (ERRW): statistical view

- Given reversible Markov Chain $P_x$ with transition probability $x_{ij}/x_i$ from $i$ to $j$, with unknown random vector $x$, how can we estimate $x$?
- Bayesian approach: assume prior on $x$ is $\mu^{i_0,a}$ and run Markov Chain $P^x$, then law is the one of ERRW $\mathbb{P}^{i_0,a}$ by theorem above.
- Hence, the posterior distribution after $n$ first steps is given by $\mu^{Z(n),X_n}$.
- Thus prior and posterior are conjuguate priors.
- (Diaconis and Rolles, 2006) $Z(n)$ is a minimal sufficient statistic for the model, also provide method of simulation of the posterior.

# Edge Reinforced Random Walks (ERRW): how to find the limit measure? (I)

Let us do Bayesian statistics again: assume that $\mu^{i_0,a}$ has an integrable smooth density $\varphi^{i_0,a}$ w.r.t $dx = \prod_{e\in E\setminus\{e_0\}} dx_e$ (for arbitrary $e_0 \in E$) on the simplex $\mathcal{L}_1 = \{\sum x_e = 1,\ x_e > 0\}$, then

$$\mathbb{P}^{i_0,a}(X \in [x, x+dx] | X_0 = i_0, \ldots, X_n = i_n = j_0)$$
$$= \frac{\varphi^{i_0,a}(x)\, dx}{\mathbb{P}^{i_0,a}(X_0 = i_0, \ldots, X_n = i_n = j_0)} \frac{\prod_{e\in E} x_e^{n_e}}{\prod_{i\in V} x_i^{v_i - \delta_{j_0}(i)}}.$$

Therefore

$$\varphi^{j_0,\alpha}(x) = \varphi^{i_0,a}(x) \frac{\prod_{e\in E} x_e^{n_e}}{\prod_{i\in V} x_i^{v_i-\delta_{j_0}(i)}} \frac{\gamma(j_0,\alpha)}{\gamma(i_0,a)}$$
$$= \frac{\varphi^{i_0,a}(x)}{\gamma(i_0,a)} \left( \frac{\prod_{e\in E} x_e^{a_e}}{\prod_{i\in V} x_i^{\frac{a_i-\delta_{i_0}(i)}{2}}} \right)^{-1} \left( \frac{\prod_{e\in E} x_e^{a_e+n_e}}{\prod_{i\in V} x_i^{\frac{a_i+n_i-\delta_{j_0}(i)}{2}}} \right) \gamma(j_0,\alpha)$$

# Edge Reinforced Random Walks (ERRW): how to find the limit measure? (II)

Again we can show, using Stirling's approximation, that, letting $n = \sum_{e\in E} \alpha_e$, $\beta = \alpha/n$, we have

$$\gamma(j_0,\alpha) \underset{\forall e\in E,\, \alpha_e\to\infty}{\sim} \sqrt{n}^{|E|-1} \sqrt{2\pi}^{|V|-|E|} \frac{\prod_{e\in E}\sqrt{\beta_e}}{\sqrt{\beta_{j_0}}} \frac{1}{(\eta_\beta(\beta))^n},$$

where

$$\eta_\beta(y) = \frac{\prod_{e\in E} y_e^{\beta_e}}{\prod_{i\in V} y_i^{\beta_i/2}}.$$

Therefore, if $n\to\infty$, $\beta\to x$, $\varphi^{j_0,\alpha}(y)$ is of the order of

$$\frac{\varphi^{i_0,a}(y)}{\gamma(i_0,a)} \frac{\prod_{i\in V} y_i^{\frac{a_i-\delta_{i_0}(i)}{2}}}{\prod_{e\in E} y_e^{a_e} x_e^{-1/2}} \sqrt{\frac{y_{j_0}}{x_{j_0}}} \sqrt{n}^{|E|-1} \sqrt{2\pi}^{|V|-|E|} \left(\frac{\eta_\beta(y)}{\eta_\beta(\beta)}\right)^n. \tag{2.1}$$

# Edge Reinforced Random Walks (ERRW): how to find the limit measure? (III)

Now

$$\log\left(\frac{\eta_\beta(y)}{\eta_\beta(\beta)}\right) = -\frac{1}{4}Q_\beta(y-\beta) + O(\|y-\beta\|_\infty^3),$$

where

$$Q_\beta(y) = \sum_{i,j:j\sim i} \beta_{ij}\left(\frac{y_{ij}}{\beta_{ij}} - \frac{y_i}{\beta_i}\right)^2 = 2\sum_{\{i,j\}\in E}\frac{y_{ij}^2}{\beta_{ij}} - \sum_{i\in V}\frac{y_i^2}{\beta_i}.$$

Show that, if $\mathcal{L}_0 = \{\sum x_e = 0\}$, $dy = \prod_{e\in E\setminus\{e_0\}} dy_e$,

$$\int_{\mathcal{L}_0} \exp\left(-\frac{1}{4}Q_\beta(y)\right) dy = \left(\prod_{e\in E}\sqrt{\beta_e}\right)\left(\prod_{i\in V}\sqrt{\beta_i}\right)\frac{2^{\frac{|E|+|V|-3}{2}}\sqrt{\pi}^{|E|-1}}{\sqrt{D(\beta)}},$$

where $D(y) = \sum_{T\in\mathcal{T}}\prod_{e\in T} y_e$, $\mathcal{T}$ set of (non-oriented) spanning trees of $G$.

# Edge Reinforced Random Walks (ERRW): how to find the limit measure? (IV)

This implies by (2.1) that

$$\begin{aligned}1 &= \lim_{n\to\infty,\, \beta\to x\in\mathcal{L}_1} \int_{\mathcal{L}_1} \varphi^{j_0,\alpha}(y)\, dy \\ &= \frac{\varphi^{i_0,a}(x)}{\gamma(i_0,a)} \frac{2^{|V|-3/2}\sqrt{\pi}^{|V|-1}}{\sqrt{x_{i_0} D(x)}} \frac{\prod_{i\in V} x_i^{\frac{a_i+1}{2}}}{\prod_{e\in E} x_e^{a_e-1}}\end{aligned}$$

which yields

$$\varphi^{i_0,a}(x) = C\gamma(i_0,a)\sqrt{x_{i_0}}\, \frac{\prod_{e\in E} x_e^{a_e-1}}{\prod_{i\in V} x_i^{\frac{a_i+1}{2}}},$$

with

$$C = \frac{2^{3/2-|V|}}{\sqrt{\pi^{|V|-1}}}.$$

# Edge Reinforced Random Walks (ERRW): Limit measure (Diaconis and Coppersmith, 1986, Keane and Rolles, 2000)

Theorem

- $(Z_e(n))_{n\in\mathbb{N}}$ *converges a.s. to a random vector* $X = (X_e)_{e\in E}$
- *Conditionally on* $x$, *ERRW is a reversible Markov chain* $P^x$ *with jump probability* $x_{ij}/x_i$ *from* $i$ *to* $j$, $x_i = \sum_{k\sim i} x_{ik}$.
- $X$ *has the following density w.r.t to measure* $\prod_{e\in E\setminus\{e_0\}} dx_e$ *on the simplex* $\{\forall e \in E,\ x_e > 0\ \sum_{e\in E} x_e = 1\}$

$$C\gamma(i_0, a)\sqrt{x_{i_0}}\frac{\prod_{e\in E} x_e^{a_e - 1}}{\prod_{i\in V} x_i^{\frac{1}{2}a_i}}\sqrt{D(x)}.$$

*Recall that*

$$\gamma(i_0, a) = \frac{\prod_{i\in V} \Gamma\left(\frac{1}{2}(a_i + 1 - \delta_{i_0}(i))\right) 2^{\frac{1}{2}(a_i - \delta_{i_0}(i))}}{\prod_{e\in E} \Gamma(a_e)},$$

# Edge Reinforced Random Walks (ERRW): Limit measure (Diaconis and Coppersmith, 1986, Keane and Rolles, 2000)

and

$$C = \frac{2^{3/2-|V|}}{\sqrt{\pi^{|V|-1}}}, \ D(y) = \sum_{T\in\mathcal{T}} \prod_{e\in T} y_e,$$

where $\mathcal{T}$ is the set of (non-oriented) spanning trees of $G$.

- Obtained by Keane and Rolles (2000) by different means: probability of a path $\mathbb{P}^{a,i_0}(X_0 = i_0, \ldots, X_n = i_n = j_0)$ only depends on the local time, compute the probability of reaching a given vertex $j_0$ and local time $(n_e)_{e\in E}$, by summing over all possible paths
- By-product of that approach: obtain an interpretation of the spanning trees which appear in the formula, as last/first exit trees of the walk. Interesting connection with work of Angel, Crawford and Kozma (2014) on recurrence of ERRW.
- As far as I know, the approach in this lecture for finding the limit measure is new.

## How to show the limit measure is correct? (Sabot-T. 2021, for *-ERRW) (I)

Let $\varphi$ be a smooth function on $\mathcal{L}_1$ whose support is compact and has empty intersection with $\cup_{e\in E}\{x : x_e = 0\}$. Let, for all $i \in V$, $\alpha \in (0,\infty)^E$,

$$\Psi(i,\alpha)(\varphi) = \int_{\mathcal{L}_1} \varphi(y)\mu^{i,\alpha}(dy)$$

and prove that

$$\mathbb{E}_{i_0}^{(\alpha)}(\varphi(Y)) = \Psi(i_0,\alpha)(\varphi).$$

The process $(X_n, Z(n))$ is a Markov process on $V \times (0,\infty)^E$ with generator

$$Lg(i,\alpha) = \sum_{j\sim i} \frac{\alpha_{i,j}}{\alpha_i}\left(f(j,\alpha+\mathbb{1}_{\{i,j)}) - f(i,\alpha)\right).$$

We have

$$L\Psi = 0. \text{ (Exercise)}$$

## How to show the limit measure is correct? (Sabot-T. 2021, for *-ERRW) (II)

This implies that $\Psi(X_n, Z(n))(\varphi)$ is a martingale, and therefore that

$$\Psi(i_0, \alpha)(\varphi) = \mathbb{E}_{i_0}^{(\alpha)}(\Psi(X_n, Z(n))(\varphi))$$

The next aim is to prove that

$$\lim_{n\to\infty} \Psi(X_n, Z(n))(\varphi) = \varphi(Y) \text{ a.s. },$$

where $Y = \lim_{n\to\infty} Z(n)/n$, which will imply the result by dominated convergence, which can be shown by an asymptotic technique as before.

## How to show the limit measure is correct, if we know it is a probability measure?

- Sample $x$ according to $\mu^{i_0,a}$ as in the formula.
- Conditionally on $x$, probability of a path $(i_0, \ldots, i_{n+1})$ is

$$\frac{\prod_{e\in E} x_e^{n_e+\delta_{\{i_n,i_{n+1}\}}(e)}}{\prod_{i\in V} x_i^{v_i}}$$

  where $n_e$ (resp. $v_i$) is the number of crossings (resp. visits) of edge $e$ (resp. site $i$) of $(i_0, \ldots, i_n)$.
- Hence the annealed probability of a path is

$$C\gamma(i_0,a)\int \frac{\prod_{e\in E} x_e^{n_e+\delta_{i_n i_{n+1}}(e)}}{\prod_{i\in V} x_i^{v_i}} d\mu^{i_0,a}(x) = \frac{\gamma(i_0,a)}{\gamma(i_{n+1},\alpha+\delta_{i_n i_{n+1}})},$$

  where $\alpha = (a_e + n_e)_{e\in E}$.
- Thus conditional probability $(i_0, \ldots, i_{n+1})$ knowing $(i_0, \ldots, i_n)$

$$\frac{\gamma(i_n,\alpha+\delta_{\{i_n,i_{n+1}\}})}{\gamma(i_{n+1},\alpha)} = \frac{\alpha_{\{i_n,i_{n+1}\}}}{\alpha_{i_n}}.$$

# Early results on Edge-Reinforced random walk ('86-'09)

- Using partial exchangeability (Diaconis and Freedman'80) ERRW is a Random Walk in Random Environment (RWRE)
- Explicit computation of mixing measure: Coppersmith-Diaconis '86, Keane-Rolles '00
- Pemantle '88: recurrence/transience phase transition on trees:
  - Root the tree at $i_0$ for simplicity.
  - Between two visits to each vertex, once an edge is crossed the walk comes back through it.
  - Hence, independently at each vertex, Pólya urn with initial number of balls $((a_{ij} + \delta_{\{j \text{ father of } i_0\}})/2)_{j\sim i}$.
  - Hence the environment is independent Dirichlet at each vertex $i$: Random Walk in (independent) Random Environment (RWRE)
- Merkl Rolles '09: recurrence on a $2d$ graph (but not $\mathbb{Z}^2$)

# ERRW and statistical physics: ERRW $\longleftrightarrow$ VRJP (I)

Let $(W_e)_{e\in E}$ be conductances on edges, $W_e > 0$.
VRJP $(Y_s)_{s\geqslant 0}$ is a continuous-time process defined by $Y_0 = i_0$ and, if $Y_s = i$, then, conditionally to the past,

$$Y \text{ jumps to } j \sim i \text{ at rate } W_{i,j} L_j(s),$$

with

$$L_j(s) = 1 + \int_0^s \mathbb{1}_{\{Y_u = j\}} du.$$

Proposed by Werner and first studied **on trees** by Davis, Volkov ('02,'04).

# ERRW and statistical physics: ERRW $\longleftrightarrow$ VRJP (II)
## Random conductances $(W_e)_{e\in E}$

Theorem (T. '11, Sabot, T. '15)

*ERRW* $(X_n)_{n\in\mathbb{N}}$ *with weights* $(\alpha_e)_{e\in E}$

$\overset{\text{"law"}}{=}$ *VRJP* $(Y_t)_{t\geqslant 0}$ *with conductances* $W_e \sim \Gamma(\alpha_e)$ *indep.*

*(at jump times)*

- Similar equivalence applies to any linearly reinforced RW on its continuous time version (initially proved for VRRW, T'. 11)

# Proof of ERRW $\longleftrightarrow$ VRJP (I)
## Rubin construction : continuous equivalent of ERRW

Similar to continuous-time version of discrete-time Markov chain

Clocks at each edge:

- $(\zeta_i^e)_{e\in E, i\in\mathbb{N}}$ collection of i.i.d variables, $Exp(1)$ distributed.
- Alarms at each edge $e \in E$, at times

$$V_k^e := \sum_{i=0}^{k} \frac{\zeta_i^e}{\alpha_e + i}, \; k \in \mathbb{N} \cup \{\infty\}.$$

Process $(\tilde{X}_t)_{t\geqslant 0}$ starting from $i_0 \in V$:

- Clock $e$ only runs when $(\tilde{X}_t)_{t\geqslant 0}$ adjacent to $e$.
- Alarm $e$ rings $\implies$ $\tilde{X}_t$ traverses it.

Then $(\tilde{X}_t)_{t\in\mathbb{R}_+}$ (at jump times) $\overset{\text{"law"}}{=}$ $(X_n)_{n\geqslant 0}$.

# Proof of ERRW $\longleftrightarrow$ VRJP (II)
## Yule process: a result of D. Kendall ('66)

For all $e \in E$, $t \geqslant 0$, let

$$N_t^e := \text{nb. of alarms at time } t \text{ for } e.$$

Then $\exists W_e \sim \text{Gamma}(\alpha_e)$ s.t., conditionally to $W_e$,

$$N_\cdot^e \text{ increases between } t \text{ and } t + dt \text{ with prob. } W_e e^t \, dt.$$

Consequences on Rubin construction:

- Let $T_x(t)$ time spent in $x \in V$ at time $t$
- Then, conditionally to $W_e$, $e \in E$, and to the past $\leqslant t$, if $\tilde{X}_t = x$, $\tilde{X}$ jumps to $y \sim x$ between $t$ and $t + dt$ with prob. $W_{xy} e^{T_x(t)+T_y(t)} \, d(T_x(t)) = W_{xy} L_y(t) d(L_x(t))$, where

$$L_z(t) := e^{T_z(t)}.$$

# VRJP: three timescales (I)

Jump rates from $i$ to $j$

- Initial timescale process $Y$, with local time $L$ :

$$W_{ij} L_j(t), \text{with} \quad L_j(s) = 1 + \int_0^s \mathbb{1}_{\{Y_u = j\}} du.$$

- Reversible timescale process $Z$, with local time $T$:

$$W_{ij} e^{T_i(t) + T_j(t)}, \text{with} \quad T_j(s) = \int_0^s \mathbb{1}_{\{Z_u = j\}} du.$$

- Exchangeable timescale process $X$:

$$\frac{1}{2} W_{ij} \sqrt{\frac{1 + \ell_j}{1 + \ell_i}}, \text{with} \quad \ell_j(s) = \int_0^s \mathbb{1}_{\{X_u = j\}} du.$$

# VRJP: three timescales (II)

Proof: Change "clocks" at all sites:

- $Z$: $T_j = \log L_j$, or $L_j = e^{T_j}$ (already appears in the proof of ERRW $\longleftrightarrow$ VRJP)
- $X$: $\ell_j = L_j^2 - 1$, or $L_j = \sqrt{1 + \ell_j}$.

Then

$$W_{ij} L_j dL_i = \frac{1}{2} W_{ij} \sqrt{\frac{1+\ell_j}{1+\ell_i}} d\ell_i = e^{T_i + T_j} dT_i.$$

# VRJP $X$ conditioned on past up to time $t$

**Lemma**

*$VRJP(i_0, W)$ $X$, conditioned on a path up to time $t$ with local time $s = (s_i)_{i \in V}$ and end position $j_0$, is a $VRJP(j_0, W^s)$ after time $t$, with time change $\ell'_i$, with*

$$W^s_{ij} = W_{ij}\sqrt{1+s_i}\sqrt{1+s_j}, \;\; \ell'_i = \frac{\ell_i - s_i}{1+s_i}.$$

**Proof.**

After that conditioning and after time $t$, $VRJP(W, i_0)$ $X$ jumps from $i$ to $j$ at a rate

$$\frac{1}{2}W_{ij}\sqrt{\frac{1+\ell_j}{1+\ell_i}}d\ell_i = \frac{1}{2}W_{ij}\sqrt{\frac{1+s_j}{1+s_i}}\sqrt{\frac{1+\frac{\ell_j - s_j}{1+s_j}}{1+\frac{\ell_i - s_i}{1+s_i}}}d\ell_i$$

$$= \frac{1}{2}W_{ij}\sqrt{1+s_i}\sqrt{1+s_j}d\ell'_i = \sqrt{\frac{1+\ell'_j}{1+\ell'_i}}d\ell'_i.$$

# VRJP: Probability of a given path (Sabot-T.,2016)

Notation. $i_0 \stackrel{v_0}{\to} i_1 \dots \stackrel{v_{n-1}}{\to} i_n = j_0, t$ denotes the event that, at time $t$, the walk spends time in $[v_0, v_0 + dv_0]$ at site $i_0$, then jumps to $v_1$ ... until it jumps to $i_n = j_0$, at which it spends the rest of the time until time $t$, and let $s_i$ be the total time spent at $i$ at that time.

Lemma

$$\mathbb{P}^{i_0,W}\left(i_0 \stackrel{v_0}{\to} i_1 \dots \stackrel{v_{n-1}}{\to} i_n = j_0, t\right)$$
$$= \frac{\exp\left(-\sum_{\{i,j\}\in E} W_{ij}\left(\sqrt{1+s_i}\sqrt{1+s_j}-1\right)\right)}{\prod_{i\neq j_0}\sqrt{1+s_i}} \prod_{k=0}^{n-1} \frac{W_{i_k i_{k+1}}}{2} dv_{i_k}.$$

# VRJP: Probability of a given path:proof (I)

Let, for all $\psi \in \mathbb{R}^V$, $i \in V$, $t \geqslant 0$,

$$F(\psi) = \sum_{\{i,j\} \in E} W_{ij} \psi_i \psi_j, \;\; G_i(t) = \prod_{j \neq i} (1 + \ell_j(t))^{-1/2}.$$

First note that the probability, for the time-changed VRJP $X$, of holding at a site $v \in V$ on a time interval $[t_1, t_2]$ is

$$\exp\left( - \int_{t_1}^{t_2} \sum_{j \sim X_t} \frac{W_{X_t,j}}{2} \frac{\sqrt{1 + \ell_j(t)}}{\sqrt{1 + \ell_{X_t}(t)}} dt \right)$$
$$= \exp\left( - \int_{t_1}^{t_2} d\left( F(\sqrt{1 + \ell(t)}) \right) \right).$$

# VRJP: Probability of a given path:proof (II)

Second, conditionally on $(X_u, u \leqslant t)$, the probability that $X$ jumps from $X_t = i$ to $j$ in the time interval $[t, t + dt]$ is

$$W_{ij}\sqrt{\frac{1+\ell_j(t)}{1+\ell_i(t)}}\,dt = W_{ij}\frac{G_j(t)}{G_i(t)}\,dt.$$

Therefore the product of jump probabilities is

$$\prod_{i=1}^{n} W_{x_{i-1}x_i}\frac{G_{x_i}(t_i)}{G_{x_{i-1}}(t_i)}\,dt_i = \prod_{i=1}^{n}\frac{W_{x_{i-1}x_i}}{2}\,dt_i,$$

where we use that $G_{x_i}(t_i) = G_{x_i}(t_{i+1})$, since $X$ stays at site $x_i$ on the time interval $[t_i, t_{i+1}]$.

# Consequence of exchangeability of VRJP

### Theorem

*VRJP$(i_0, W)$ $X$ is a mixture of Markov Jump Processes (MJP) $P^u$ with jump rate from $i$ to $j$*

$$\frac{1}{2} W_{ij} e^{u_j - u_i}.$$

### Proof.

The probability of a given path for $X$ only depends on the local time, initial and final position, which implies partial exchangeability (Zeng, 2013).
Hence

$$U_i = \lim_{t\to\infty} \frac{1}{2} \log \ell_i(t) - \frac{1}{|V|} \sum_{j\in V} \lim_{t\to\infty} \frac{1}{2} \log \ell_j(t)$$

exists and conditionally on $U$, the jump rate from $i$ to $j$ is $W_{ij} e^{U_j - U_i}/2$. $\square$

# Bayesian approach to find limit measure of VRJP (I)

- Assume that the law $\mu^{i_0,W}$ of $U$ under $\mathbb{P}^{i_0,W}$ has a smooth integrable density measure $\varphi^{i_0,W}$w.r.t. Lebesgue measure $du$ on the simplex $\mathcal{L}_0 = \{u : \sum_{e\in E} u_e = 0\}$.
- Note that the probability of $i_0 \overset{v_0}{\to} i_1 \dots \overset{v_{n-1}}{\to} i_n = j_0, t$ under $P^u$ is

$$\exp\left(-\sum_{i,j\sim i} \frac{1}{2} W_{ij} e^{u_j - u_i} s_i\right) e^{u_{j_0} - u_{i_0}} \prod_{k=0}^{n-1} \frac{W_{i_k i_{k+1}}}{2} dv_{i_k}.$$

- Notation $[(u_i)_{i\in V}] = \left(u_i - \frac{1}{|V|}\sum_{j\in V} u_j\right)_{i\in V}$.
- After conditioning on path up to time $t$, we have new weights and new potentials

$$W_{ij}^s = W_{ij}\sqrt{1+s_i}\sqrt{1+s_j}, U' = U - \left[\frac{1}{2}\log(1+s)\right].$$

# Bayesian approach to find limit measure of VRJP (II)

Hence

$$\mathbb{P}^{i_0,W}(U \in [u,u+du] | i_0 \stackrel{v_0}{\to} i_1 \dots \stackrel{v_{n-1}}{\to} i_n = j_0, t)$$
$$= \frac{\varphi^{i_0,W}(u) \exp\left(-\sum_{i,j\sim i} \frac{1}{2} W_{ij} e^{u_j - u_i} s_i\right) e^{u_{j_0} - u_{i_0}}}{\exp\left(-\sum_{\{i,j\}\in E} W_{ij}\left(\sqrt{1+s_i}\sqrt{1+s_j} - 1\right)\right)} \prod_{i \neq j_0} \sqrt{1+s_i}\, du$$
$$= \varphi^{j_0,W^s}\left(u - \left[\frac{\log(1+s)}{2}\right]\right) du$$

Therefore

$$\varphi^{j_0,W^s}\left(u - \left[\frac{\log(1+s)}{2}\right]\right) \tag{3.1}$$
$$= \frac{\varphi^{i_0,W}(u) \exp\left(-\sum_{i,j\sim i} \frac{1}{2} W_{ij} e^{u_j - u_i} s_i\right) e^{u_{j_0} - u_{i_0}}}{\exp\left(-\sum_{\{i,j\}\in E} W_{ij}\left(\sqrt{1+s_i}\sqrt{1+s_j} - 1\right)\right)} \prod_{i \neq j_0} \sqrt{1+s_i}.$$

# Bayesian approach to find limit measure of VRJP (III)

Applying the result for $s$ s.t. $1 + s_i = e^{2u_i}$, $i \in V$, yields

$$\varphi^{j_0, W^s}(0) = \varphi^{i_0, W}(u) \exp\left(\frac{1}{2}\sum_{i, j \sim i} W_{ij}(e^{u_j - u_i} - 1)\right) e^{-u_{i_0}}. \quad (3.2)$$

On the other hand, let

$$\eta_{W,h}(u) = \exp\left(-\sum_{i,j\sim i} \frac{1}{2} W_{ij} e^{u_j - u_i}(h-1)\right).$$

Then, if $s_i = h - 1$ and $j_0 = i_0$, we deduce again from (3.1) that

$$\varphi^{i_0, Wh}(u) = \varphi^{i_0, W}(u)\left(\frac{\eta_{W,h}(u)}{\eta_{W,h}(0)}\right) h^{\frac{|V|-1}{2}}.$$

# Bayesian approach to find limit measure of VRJP (IV)

Now

$$\log\left(\frac{\eta_{W,h}(u)}{\eta_{W,h}(0)}\right) = -(h-1)(Q(u)+O(\|u\|^3)),$$

where

$$Q(u) = \frac{1}{4}\sum_{i,j\sim i}(u_j-u_i)^2.$$

Using that

$$\int_{\mathcal{L}_0} \exp\left(-sQ(u)\right)\, du = \frac{(2\pi s^{-1})^{(|V|-1)/2}}{|V|\sqrt{D(W,0)}},$$

where

$$D(W,u) = \sum_{T\in\mathcal{T}}\prod_{\{i,j\}\in T} W_{\{i,j\}}e^{u_i+u_j},$$

we can compute $\varphi^{i_0,W}(0)$ since

$$1 = \int_{\mathcal{L}_0}\varphi^{i_0,Wh}(u)\,du = \varphi^{i_0,W}(0)\frac{(2\pi)^{(|V|-1)/2}}{|V|\sqrt{D(W,0)}}.$$

## VRJP $\longleftrightarrow$ SuSy hyperbolic sigma model in QFT (I)
## Fixed conductances $(W_e)_{e\in E}$, $G$ finite (Sabot-T.'15)

The measure $\mu^{i_0,W}(du)$ has density on $\mathcal{L}_0=\{(u_i),\ \sum u_i=0\}$

$$\frac{N}{(2\pi)^{(N-1)/2}}e^{u_{i_0}}e^{-H(W,u)}\sqrt{D(W,u)},$$

where

$$H(W,u)=2\sum_{\{i,j\}\in E}W_{i,j}\sinh^2((u_i-u_j)/2).$$

and

$$D(W,u)=\sum_{T\in\mathcal{T}}\prod_{\{i,j\}\in T}W_{\{i,j\}}e^{u_i+u_j}.$$

# VRJP $\longleftrightarrow$ SuSy hyperbolic sigma model in QFT (II)

## Fixed conductances $(W_e)_{e\in E}$, $G$ finite (Merkl-Rolles-T.'19)

- $\mathcal{Q}^{i_0,W}(du)$ marginal of Gibbs "measure" on supermanifold extension $\mathbb{H}^{2|2}$ of hyperbolic plane with action $A_W(v,v) = \sum_{i,j} W_{ij}(v_i - v_j, v_i - v_j)$, taken in horospherical coordinates after integration over fermionic variables.

- Merkl-Rolles-T.'19: Other variables in extension of SuSy model arise on two different time scales as limits of
  - local times on logarithmic scale
  - rescaled fluctuations of local times
  - rescaled crossing numbers
  - last exit trees of the walk (tree version of fermionic variables)
- Bauerschmidt-Helmuth-Swan '19 (AP and AIHP): very nice interpretation of in terms of Brydges-Fröhlich-Spencer-Dynkin isomorphism for the supersymmetric field.

## VRJP $\longleftrightarrow$ random Schrödinger (Sabot-T.-Zeng '15) (I)

Let, for all $i \in V$,

$$\beta_i = \frac{1}{2} \sum_{j \sim i} W_{ij} e^{u_j - u_i} + \delta_{i_0}(i)\gamma,$$

$\gamma \sim \Gamma(1/2)$ indep. of $u$.

- $\forall i \neq i_0$, $\beta_i =$ jump rate from $i$
- $\beta$ field 1-dependent: $\beta_{|V_1}$ and $\beta_{|V_2}$ are independent if $dist_{\mathcal{G}}(V_1, V_2) \geqslant 2$.
- On $\mathbb{Z}^d$ with $W_{ij} = W$ constant, $(\beta_i)_{i \in V}$ translation-invariant
- The marginals $\beta_i$ are such that $(2\beta_i)^{-1}$ have inverse Gaussian law.

# VRJP $\longleftrightarrow$ random Schrödinger: Range and law of $\beta$ (II)

- $V$ finite
- $\Delta = (\Delta_{i,j})_{i,j \in V}$ discrete Laplacian, letting $W_i := \sum_{j \sim i} W_{i,j}$,

$$\Delta_{i,j} := \begin{cases} W_{i,j}, & \text{if } i \sim j,\ i \neq j \\ -W_i, & \text{if } i = j \end{cases}$$

- $H_\beta := -\Delta + 2\beta$, $W$ diagonal with coefficients $(W_i)_{i \in V}$.
- $H_\beta > 0$ (positive definite): $\implies$ $(H_\beta)^{-1}$ has positive entries.
- $\beta = (\beta_i)_{i \in V}$ has distribution

$$\nu^W(d\beta) = \sqrt{\frac{2}{\pi}}^{|V|} \mathbb{1}_{\{H_\beta > 0\}} \frac{e^{\sum_{i \in V}(W_i/2 - \beta_i)}}{\sqrt{|H_\beta|}} \prod_{i \in V} d\beta_i.$$

# VRJP $\longleftrightarrow$ random Schrödinger: Retrieve $u$ from $\beta$ (III)

- Set $G = (H_\beta)^{-1}$.
- Then

$$\beta_i = \frac{1}{2}\sum_{j\sim i} W_{ij} e^{u_j - u_i}, \;\; i \neq i_0$$
$$\iff H_\beta(e^{u_\cdot})(i) = (-\Delta + 2\beta)(e^{u_\cdot})(i) = 0, \;\; i \neq i_0$$
$$\iff e^{u_i} = \frac{G(i_0, i)}{G(i_0, i_0)}, \;\; i \in V$$

where $(u_i)_{i\in V}$ defined above and follows the law $\mathcal{Q}_{i_0}^W(du)$.

- Hence, time-changed VRJP starting from $i_0$ mixture of Markov jump processes with jump rate

$$\frac{1}{2} W_{i,j} e^{u_j - u_i} = \frac{1}{2} W_{i,j} \frac{G(i_0, j)}{G(i_0, i)}$$

# ERRW/VRJP and statistical physics: implications

Using link with QFT and localisation/delocalisation results of Disertori, Spencer, Zirnbauer '10 :

**Theorem (ST'15, Angel-Crawford-Kozma'14, $G$ bded degree)**

*ERRW (resp.VRJP) is positive recurrent at strong reinforcement, i.e. for $a_e$ (resp. $W_e$) uniformly small in $e \in E$.*

**Theorem (ST'15, Disertori-ST'15, $G = \mathbb{Z}^d$, $d \geqslant 3$)**

*ERRW (resp. VRJP) is transient at weak reinforcement, i.e. for $a_e$ (resp. $W_e$) uniformly large in $e \in E$.*

Using link with Random Schrödinger operator:

**Theorem (Sabot-Zeng '19, Sabot -19, Merkl-Rolles '09)**

*ERRW with constant weights $a_e = a$ (resp. $W_e = W$) is recurrent in dimension 2.*

**Theorem (Poudevigne'19)**

*Increasing initial weights of ERRW and VRJP makes them more transient (unique phase transition).*

# *-Edge-Reinforced Random Walk motivation : Reversible $k$-dependent Markov chains

- $(Y_i)$ $k$-dependent Markov chain on $S$ finite (i.e. law of $Y_{n+1}$ depends only on $(Y_{n-k+1}, \ldots, Y_n)$).
- Equivalent to Markov chain $(X_n)$ on the (directed) de Bruijn graph $G = (V = S^k, E)$ with

$$\omega = (i_1, \ldots, i_k) \to \tilde{\omega} = (i_2, \ldots, i_{k+1})$$

  with transition rate $p(\omega, \tilde{\omega})$, and invariant measure $\pi(\omega)$.

The $k$-dependent Markov chain is called reversible if

$$(Y_1, \ldots, Y_n) \overset{law}{=} (Y_n, \ldots, Y_1).$$

as soon as $(Y_1, \ldots, Y_k) \sim \pi$ invariant measure. This is equivalent to the "modified" balance condition

$$\pi(\omega) p(\omega, \tilde{\omega}) = \pi(\tilde{\omega}^*) p(\tilde{\omega}^*, \omega^*),$$

where $\omega^*$ is the flipped $k$-string $\omega^* = (i_k, \ldots, i_1)$.

## General framework

- $G = (V, E)$ directed graph with involution $*$ on $V$ s.t.

$$(i, j) \in E \Rightarrow (j^*, i^*) \in E$$

- Let $V_0 = \{i \in V : i = i^*\}$, and $V_1$ be s.t. $V = V_0 \cup V_1 \cup V_1^*$ disjoint.
- $\alpha_{i,j} > 0$, $(i, j) \in E$ such that $\alpha_{i,j} = \alpha_{j^*,i^*}$.

We call $\star$-ERRW with initial weights $(\alpha_e)$, the discrete time process $(X_n)$ defined by

$$\mathbb{P}(X_{n+1} = j \,|\, X_k,\ k \leqslant n) = \mathbb{1}_{\{X_n \to j\}} \frac{Z_{(X_n,j)}(n))}{\sum_{l, X_n \to l} Z_{(X_n,l)}(n))}$$

where

$$Z_{(i,j)}(n) = \alpha_{i,j} + N_{i,j}(n) + N_{j^*,i^*}(n)$$

$$N_{(i,j)}(n) = \sum_{k=1}^{n} \mathbb{1}_{\{(X_{k-1},X_k)=(i,j)\}}.$$

# *-ERRW: partial exchangeability

Let

$$\mathrm{div}(z)(i) = \sum_{j, i\to j} z_{i,j} - \sum_{j, j\to i} z_{j,i}, \quad \mathrm{div} : \mathbb{R}^E \mapsto \mathbb{R}^V,$$

$$\gamma(i_0, \alpha)$$

$$= \frac{\left(\prod_{i\in V_0} \Gamma(\frac{1}{2}(\alpha_i + 1 - \mathbb{1}_{i=i_0}) 2^{\frac{1}{2}(\alpha_i - \mathbb{1}_{i=i_0})}\right)\left(\prod_{i\in V_1} \Gamma(\inf(\alpha_i, \alpha_{i^*}))\right)}{\prod_{(i,j)\in\tilde{E}} \Gamma(\alpha_{i,j})}.$$

Let $\tilde{E}$ be the set of edges quotiented by the relation $(i,j) \sim (j^*, i^*)$.

Proposition (Bacallado '11, Baccalado, Sabot and T. '21)

*Let $i_0 \in V$. If div$(\alpha) = \delta_{i_0^*} - \delta_{i_0}$, then the $\star$-ERRW starting from $i_0$ is partially exchangeable. More precisely, if $\beta_e = \alpha_e + n_e$, $n_{(i,j)}$ number of crossings of oriented edges $(i,j)$ and $(j^*, i^*)$, then*

$$\mathbb{P}^{a,i_0}(X_0 = i_0, \ldots, X_n = i_n = j_0) = \frac{\gamma(i_0, \alpha)}{\gamma(j_0, \beta)}.$$

# *-Edge Reinforced Random Walks (*-ERRW): statistical view

- Statistical analysis of molecular dynamics simulations with microscopically reversible laws.
- Two other applications, beyond Bayesian analysis of higher-order Markov chains (Bacallado, 2006):
  - Variable-order Markov chains with context set $\mathcal{C} \subseteq S \cup S^2 \cup \cdots \cup S^k$ on de Bruijn graph: $\forall (i_1, \ldots, i_\ell) \in \mathcal{C}$, transition probabilities out of $x$ and $y$ are the same whenever $x$ and $y$ both end in $(i_1, \ldots, i_\ell)$.
  - Reinforced random walk with amnesia: RW on $G = (V, E)$ defined by $V = S \cup S^2 \cup \ldots S^k$ with two types of edges: "forgetting" ones $(i_1, \ldots, i_m) \to (i_2, \ldots, i_m)$, if $m > 1$, "appending" ones $(i_1, \ldots, i_m) \to (i_1, \ldots, i_m, j)$, for each $j \in V$, if $m < k$. Can be seen as generalization of the above, by disallowing appending when word ends with subword in the context set.

# *-Edge Reinforced Random Walks (*-ERRW): results

Theorem (Bacallado, Sabot and T., 2021)

- $(Z_n(e)/n)_{n\in\mathbb{N}}$ *converges a.s. to random vector* $X$ *in* $\mathcal{L}_1 = \left\{x \in (0,\infty)^E : x_{i,j} = x_{j^*,i^*},\, div(x) = 0,\, \sum_{e\in E} x_e = 1\right\}.$
- *Conditionally on* $x$, *ERRW is a Markov chain* $P_x$ *with jump probability* $x_{ij}/x_i$ *from* $i$ *to* $j$, $x_i = \sum_{i\to k} x_{ik}$.
- *The random variable* $X$ *has the following density on* $\mathcal{L}_1$, *w.r.t* $\prod_{e\in B} dx_e$, $B$ *basis of* $\mathcal{L}_1$*:*

$$C\gamma(i_0,\alpha)\sqrt{x_{i_0}}\left(\frac{\prod_{(i,j)\in\tilde{E}} x_{i,j}^{\alpha_{i,j}-1}}{\prod_{i\in V} x_i^{\frac{1}{2}\alpha_i}}\right)\frac{1}{\prod_{i\in V_0}\sqrt{x_i}}\sqrt{D(x)}\,dx_{\mathcal{L}_1},$$

# *-Edge Reinforced Random Walks (*-ERRW): results

Recall that

$$\gamma(i_0, \alpha) = \frac{\left(\prod_{i\in V_0} \Gamma(\frac{1}{2}(\alpha_i + 1 - \mathbb{1}_{i=i_0})2^{\frac{1}{2}(\alpha_i - \mathbb{1}_{i=i_0})}\right) \left(\prod_{i\in V_1} \Gamma(\inf(\alpha_i, \alpha_{i^*}))\right)}{\prod_{(i,j)\in\tilde{E}} \Gamma(\alpha_{i,j})},$$

and

$$C = \frac{2}{\sqrt{2\pi}^{|V_0|-1}\sqrt{2}^{|V_0|+|V_1|}}, \quad D(y) = \sum_{T} \prod_{(i,j)\in T} y_{i,j}.$$

The last sum runs on spanning trees directed towards a root $j_0 \in V$ (value does not depend on the choice of the root $j_0$).

# Correspondence *-ERRW $\longleftrightarrow$ *-VRJP (I)

Let $(W_e)_{e\in E}$ be conductances on edges, $W_{ij} = W_{j^*i^*} > 0$.
The *-Vertex-Reinforced Jump Process (*-VRJP) $(Y_s)_{s\geqslant 0}$ is a continuous-time process defined by $Y_0 = i_0$ and, if $Y_s = i$, then, conditionally to the past,

$$Y \text{ jumps to } j \sim i \text{ at rate } W_{i,j} L_{j^*}(s),$$

with

$$L_j(s) = 1 + \int_0^s \mathbb{1}_{\{Y_u = j\}} du.$$

# Correspondence *-ERRW $\longleftrightarrow$ *-VRJP (II)
## Random conductances $(W_e)_{e\in E}$

Theorem (Bacallado-Sabot-T. '21)

$$\begin{aligned} &\textit{*-ERRW}\ (X_n)_{n\in\mathbb{N}}\ \textit{with weights}\ (\alpha_e)_{e\in E}, \alpha_{ij}=\alpha_{j^*i^*} \\ \overset{\text{"law"}}{=}\ &*-\textit{VRJP}\ (Y_t)_{t\geqslant 0}\ \textit{with conductances}\ W_e \sim \Gamma(\alpha_e), e\in \tilde{E}\ \textit{indep.} \\ &\textit{(at jump times)} \end{aligned}$$

Proof.
Similar to [T.'11, Sabot-T.'15], as for any linearly reinforced RW on its continuous time version. $\square$

# *-VRJP: again three timescales

Jump rates from $i$ to $j$

- Initial timescale process $Y$, with local time $L$ :

$$W_{ij}L_j^*(t), \text{with} \quad L_j(s) = 1 + \int_0^s \mathbb{1}_{\{Y_u = j\}} du.$$

- Reversible timescale process $Z$, with local time $T$:

$$W_{ij}e^{T_i(t)+T_j^*(t)}, \text{with} \quad T_j(s) = \int_0^s \mathbb{1}_{\{Z_u = j\}} du.$$

- Exchangeable timescale process $X$:

$$\frac{1}{2}W_{ij}\sqrt{\frac{1+\ell_j^*}{1+\ell_i}}, \text{with} \quad \ell_j(s) = \int_0^s \mathbb{1}_{\{X_u = j\}} du.$$

# The limiting manifold

Set $\mathcal{L}_0^W = \{(u_i)_{i\in V},\ \mathrm{div}(W^u) = 0,\ \sum_{i\in V} u_i = 0\}$.

## Proposition

*The following limit*

$$U_i = \lim_{t\to\infty} T_i(t) - t/|V|$$

*exists a.s. and* $U \in \mathcal{L}_0^W$*.*

## Proof of $U \in \mathcal{L}_0^W$.

If $X$ is at $i$, it jumps to $j$ with probability $W_{ij} d(e^{T_i(t)+T_{j^*}(t)})$ on infinitesimal time interval. Hence

$$W_{ij} e^{T_i(t)+T_{j^*}(t)}/(Z_{(ij)}(t) + Z_{(j^*i^*)}(t)) \to_{t\to\infty} 1.$$

On the other hand, by Kirchoff's law,

$$|\sum_{j:\, i\to j} (Z_{(ij)}(t) + Z_{(j^*i^*)}(t) - \sum_{k:\, k\to i} (Z_{(ki)}(t) + Z_{(i^*k^*)}(t)| \leqslant 1.$$

# Randomization of the initial local time

- Also appears in the context of self-repelling motion: T., Tóth and Valkó'12, Horváth, Tóth and Vetö'12.
- For $i_0 \in V$, consider the probability measure $\nu_{i_0}^W$ on

$$\mathcal{A} = \{(a_i) \in \mathbb{R}^V, a_{i^*} = -a_i\}$$

given by

$$\nu_{i_0}^W(da) = \frac{1}{F(W, i_0)} e^{a_{i_0^*}} e^{-\frac{1}{2}\sum_{i\to j} W_{i,j} e^{a_{j^*} - a_{i^*}}} da,$$

where $da = \prod_{i \in V_1} da_i$ and $F(W, i_0)$ normalizing constant.

## Proposition

*Let $(\alpha_e)$ be positive weights with $div(a) = \delta_{i_0^*} - \delta_{i_0}$, and $W_e \sim Gamma(\alpha_e)$ indep. Then $W^A \overset{law}{=} W$.*

Let

$$\overline{\mathbb{P}}_{i_0}^{W}(\cdot) = \mathbb{E}(\mathbb{P}_{i_0}^{W^A}(\cdot))$$

be the law of the $\star$-VRJP after expectation with respect to $A \sim \nu_{i_0}^{W}$.

## Proposition

*Let*

$$C(t) = \sum_{i \in V} (e^{T_i(t) + T_{i^*}(t)} - 1)$$

*and $Z_s = X_{C^{-1}(s)}$. Then $Z$ is partially exchangeable, and in particular there exists a random variable $U \in \mathcal{L}_0^W$ such that, conditionally on $U$, $Z$ is a Markov process with jump rates*

$$\frac{1}{2} e^{U_{j^*} - U_{i^*}} \text{ from } i \text{ to } j.$$

# Limit measure of *-VRJP

**Theorem**

*Under $\overline{\mathbb{P}}_{i_0}^W$, the random variable $U \in \mathcal{L}_0^W$ has density on $\mathcal{L}_0^W$ given by*

$$\frac{1}{\sqrt{2\pi}^{|V_0|-1} F(W, i_0)} e^{u_{i_0^*} - \sum_{i \in V_0} u_i} e^{-\frac{1}{2} \sum_{i \to j} W_{i,j} e^{u_{j^*} - u_{i^*}}} \frac{\sqrt{D(W^u)}}{\det(\mathcal{P}_{\mathcal{A}} M(W^u) \mathcal{P}_{\mathcal{A}})},$$

*with $\mathcal{P}_{\mathcal{A}}$ orthogonal projection onto $\mathcal{A}$, and*

$$D(W^u) = \sum_T \prod_{\{i,j\} \in T} W^u_{i,j},$$

*where the sum runs on* **rooted** *spanning trees of the graph, and $M(W^u)$ is the generator of the Markov jump process at rate $W^u_{i,j}$.*

# *-VRJP: Random Schrödinger version

Let $H_\beta = \beta - W$, $G_\beta = H_\beta^{-1}$.

### Theorem

*For all $\theta \in (0,\infty)^V$, $\eta \in (0,\infty)^V$, we have*

$$\prod_{i\in V_0} \theta_i \int_{\mathcal{S}} \frac{\mathbb{1}_{H_\beta>0}}{\sqrt{2\pi}^{|\mathcal{S}|}} \exp\left(-\frac{1}{2}\langle\theta, H_\beta\theta\rangle - \frac{1}{2}\langle\eta, G_\beta\eta\rangle\right)\frac{d\beta}{\sqrt{|H_\beta|}}$$
$$= \int_{\mathcal{A}} \frac{1}{\sqrt{2\pi}^{|\mathcal{A}|}} \exp\left(-\frac{1}{2}\langle e^a\theta, We^a\theta\rangle + \frac{1}{2}\langle\theta, W\theta\rangle - \langle\eta, e^a\theta\rangle\right) da.$$

When $X_0 = i_0$, the measure on $\beta$ is associated to a differentiation with respect to $\eta_{i_0}$ of a combination of the two measures above at $\eta = 0$, $\theta = 1$ on $\{i_0, i_0^*\}^c$.