Large deviations of SLE and Weil-Petersson Teichmüller space

Lecture 1

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(2018 +)

Loewner energy

SLE

Schramm-Loewner evolutions

Weil-Petersson Teichmüller space

Probabilistic

- Stochastic analysis
- Conformally invariant 2D - statistical mechanics model
- Random planar maps
- 2D - quantum gravity
- Conformal field theory

Deterministic

- quasicircles
- Quasiconformal mapping
- Geometric function theory
- Complex structures on Riemann surfaces
- Kähler geometry
- String theory

Plan for today:

I) Brownian Motion and Dirichlet energy

II) SLE and Loewner energy

III) SLE_{0+} large deviations

For tomorrow:

I) Loop energy

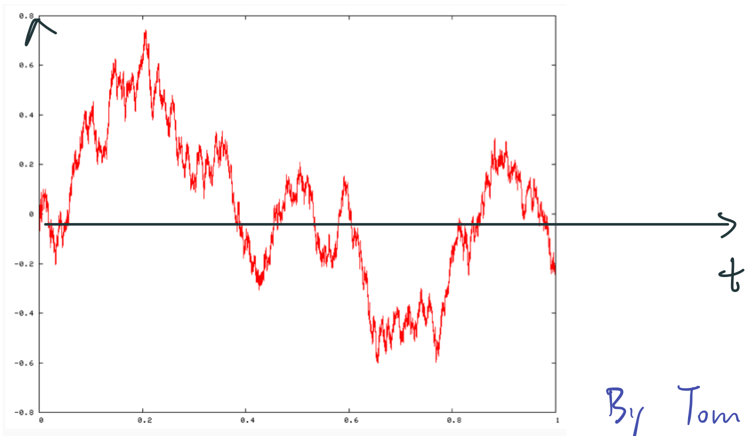
II) Weil-Petersson Teichmüller space

On Thursday:

I) Radial SLE_{∞} large deviations

II) Foliations by Weil-Petersson quasicircles

I) Brownian Motion and Dirichlet energy



By Tom Kennedy

$(k>0)$ $W := \sqrt{k}\, B$ are the unique continuous random processes which satisfy

- $W_0 = 0$
- $\mathbb{E}(W_t) = 0, \quad \forall t \in \mathbb{R}_+$
- W has independent and stationary increments
  - for $0 = t_0 < t_1 < \cdots < t_n$, $\{W_{t_0}, W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}}\}$ is an independent family
  - $W_{t_1} - W_{t_0}$ has the same distribution as $W_{t_1 - t_0}$

$\longrightarrow$ Universality of BM.

$\longrightarrow$ $W_t \sim \mathcal{N}(0, kt)$

$\rightarrow$ density of $(W_{t_0}, W_{t_1}, \cdots, W_{t_n})$

$$\propto \prod_{i=0}^{n-1} \exp\left(- \frac{(W_{t_{i+1}} - W_{t_i})^2}{k\, 2\, (t_{i+1} - t_i)} \right) dW_{t_{i+1}}$$

Physicists may write

" The density of $\sqrt{k}\, B$ is

$$\exp\left(- \frac{I(W)}{k} \right) \underbrace{DW}_{\text{Ill-defined}} \text{"}$$

Lebesgue measure on the space of function $C^0(\mathbb{R}_+, \mathbb{R})$

where

$$I(W) := \frac{1}{2} \int_0^\infty \left(\frac{dW}{dt} \right)^2 dt \qquad \text{(if } W \text{ is a.c.)}$$

$$= \frac{1}{2} \sup_{0 = t_0 < t_1 < \cdots < t_n} \left(\sum_{i=0}^{n-1} \frac{|W_{t_{i+1}} - W_{t_i}|^2}{t_{i+1} - t_i} \right)$$

is the Dirichlet energy of W.

a.s. $I(B) = \infty$

Schilder's theorem (asymptotic $k \to 0$)

$I(w)$ is the large deviation rate function of $\sqrt{k}B$ as $k \to 0$.

$$\mathrm{Prob}(\sqrt{k}B \simeq w) \underset{k\to 0}{\sim} \exp\left(-\frac{I(w)}{k}\right)$$

More precisely, for any $T < \infty$, consider $B_{[0,T]}$ as a random function in $(C_0([0,T]), \|\ \|_\infty)$

For any Borel set $A \subset C_0([0,T])$

$$-\inf_{w \in \mathring{A}} I(w) \le \varliminf_{k \to 0} k \log \mathbb{P}(\sqrt{k}B \in \mathring{A})$$

$$\le \varlimsup_{k \to 0} k \log \mathbb{P}(\sqrt{k}B \in \bar{A})$$

$$\le -\inf_{w \in \bar{A}} I(w)$$

Moreover, I is lower-semi-continuous.

(If $w_n \xrightarrow{\|\ \|_\infty} w$, $\varliminf_{n\to\infty} I(w_n) \ge I(w)$)

and I is good.

" $\forall l > 0$, $\{w \mid I(w) \le l\}$ is compact.

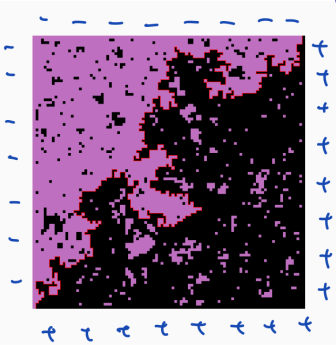
Remark: • Large deviation results depend on the topology!

• We can take $T = \infty$, but the topology is the topology for uniform convergence on compact sets.

A 2D Brownian motion (B_t^1, B_t^2)

independent

Q: Random planar curve WITHOUT self-intersection?



A simulation of interface in critical Ising lattice model, which approximates the SLE_3.

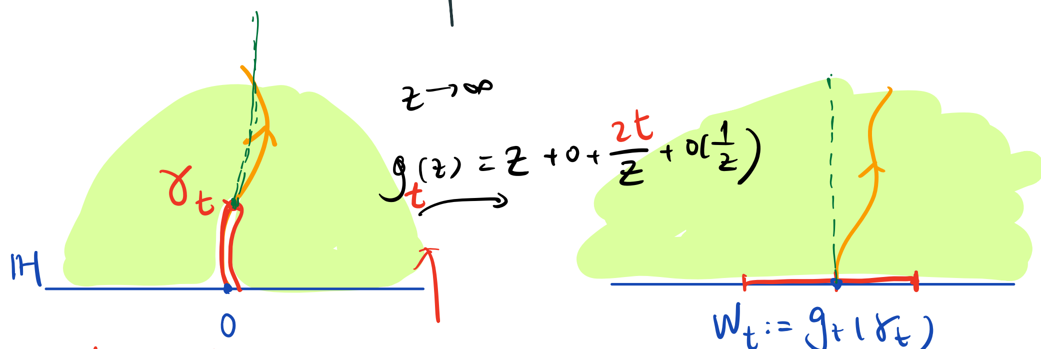
(Interfaces in 2D statistical mechanics models e.g.)

Oded Schramm introduces SLE 1999

II Schramm Loewner evolutions & Loewner energy

1) Loewner transform (Loewner 1923)

Let γ be a simple chord in $(\mathbb{H}, 0, \infty)$



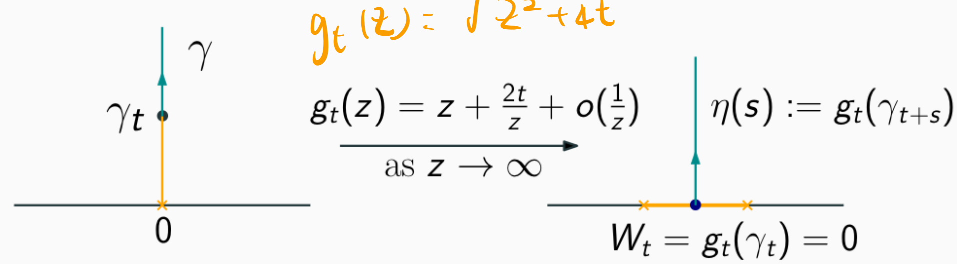
Conformal map by Riemann mapping theorem

- $W_0 = 0$, $t \mapsto W_t$ is continuous
- W is the Loewner driving function of γ
- W uniquely determines γ.

Example

If $W \equiv 0$, then $\gamma = i\mathbb{R}_+$.

$g_t(z) = \sqrt{z^2 + 4t}$



Properties:

- Additivity: The driving function of $g_t(\gamma_{[t,T]}) - W_t$

is $s \mapsto W_{t+s} - W_t$

- Scaling: The driving function of $\lambda\gamma$

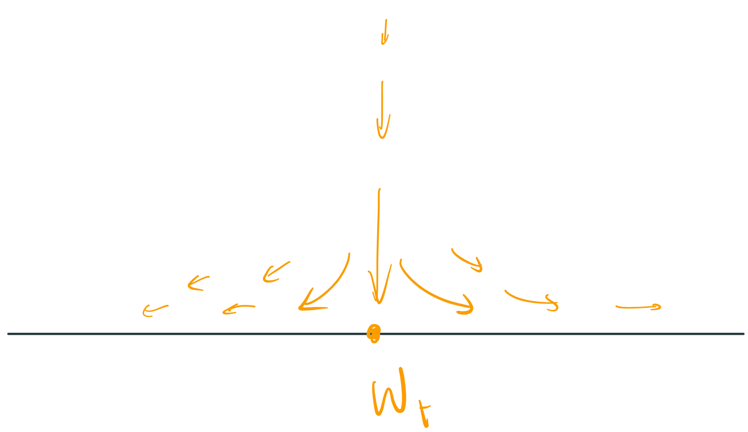
is $t \mapsto \lambda W_{\lambda^{-2}t}$, on $[0, \lambda^2 T]$

⚠ BM is invariant under these two transformations

- W_t can recover γ from W: $z \in \mathbb{H}$

$$\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z.$$

$$H_t := \{z \in \mathbb{H} \mid g_t(z) \text{ well-defined}\} = \mathbb{H} \setminus \gamma[0,t].$$



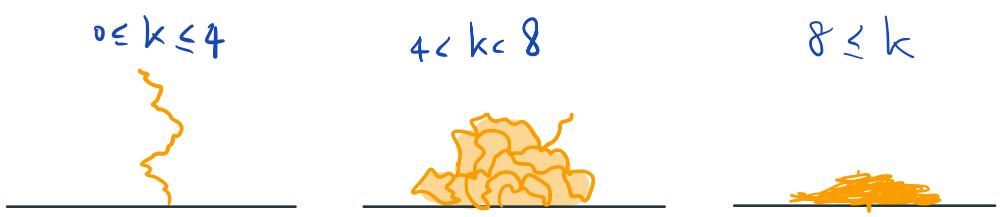
More generally, for an arbitrary continuous function W, H_t is still well-defined, but $K_t := \mathbb{H} \setminus H_t$ may not be a simple curve

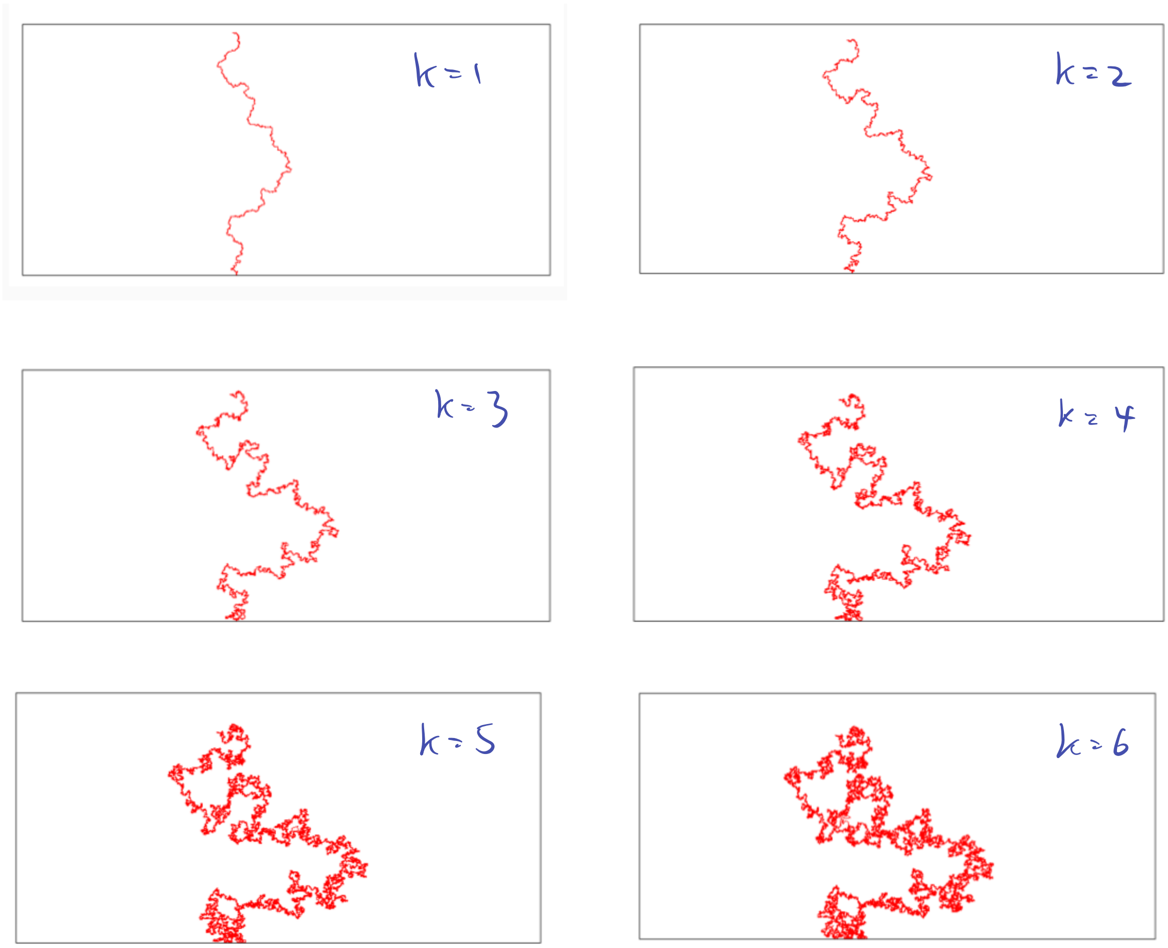
$$\Rightarrow \quad (\overset{\mathbb{H}}{\overset{\cup}{H_t}})_{t \geq 0}$$ is a decreasing family of simply connected domains which contains a neighborhood of ∞.

Definition (Schramm '99)

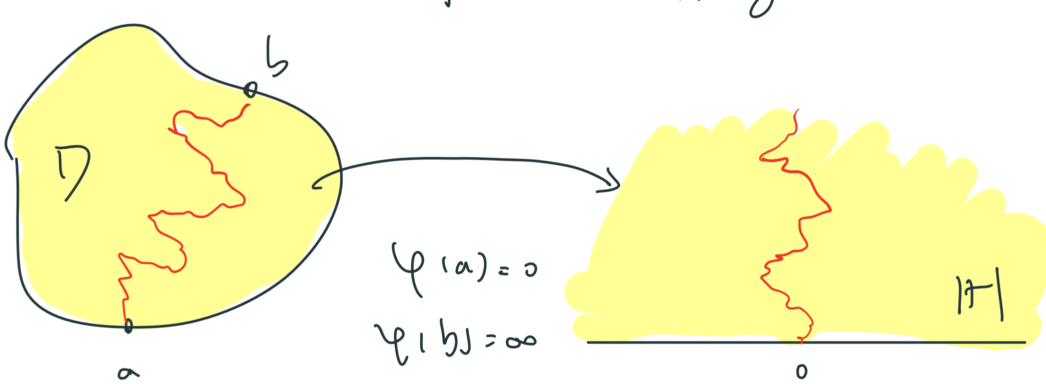
SLE_κ in $(\mathbb{H}, 0, \infty)$ is the Loewner chain driven by $\sqrt{\kappa}\, B$. $\kappa > 0$

Thm [Rohde-Schramm '05]





SLE$_\kappa$ in (D, a, b) is the pull-back of SLE$_\kappa$ in $(\mathbb{H}, 0, \infty)$ by a conformal mapping φ.

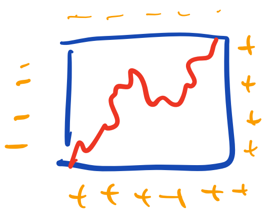


Universality of BM

$\Rightarrow$ SLE are the unique random process of non-self intersecting curve which satisfy

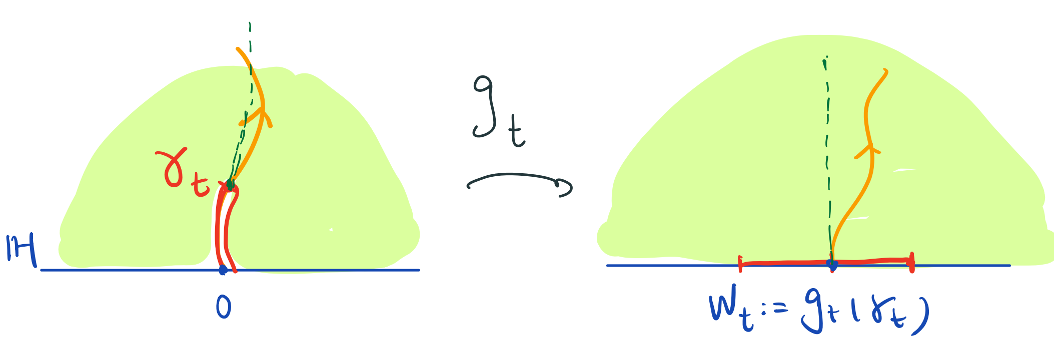
- Conformal invariance
- Domain Markov property.

$\Rightarrow$ Interfaces in stat. mechanics model



(critical Ising, GFF, Percolation ..)

Schramm. Lawler, Werner, Sheffield. Smirnov Chalkak ...



2) Loewner energy

Definition

The Loewner energy of a deterministic chord γ in $(\mathbb{H}, 0, \infty)$

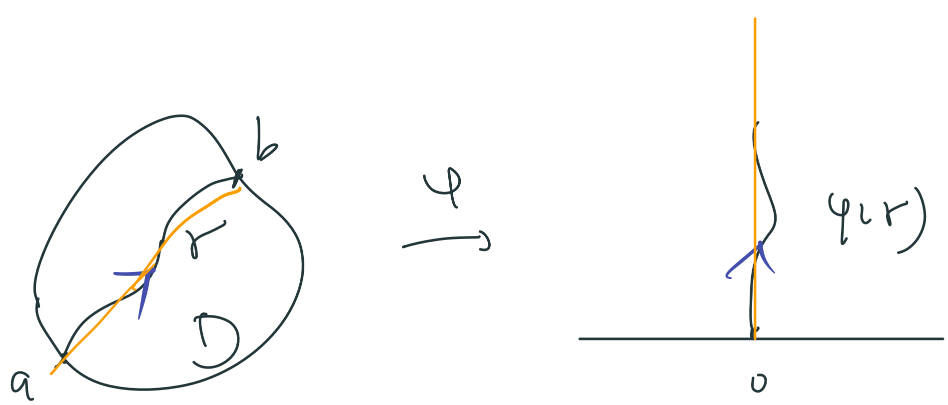
$$I_{\mathbb{H},0,\infty}(\gamma) := \frac{1}{2}\int_0^{T} (W_t')^2 \, dt \qquad \in [0,\infty]$$

T = Total capacity

where W is the Loewner driving function of γ.

Fact: If $\gamma_t \to \infty$ as $t \to \infty$ and $I_{\mathbb{H},0,\infty}(\gamma) < \infty$. Then $T = \infty$.

We will only consider this case when we have chords reaching the target point



Def

The Loewner energy of γ in (D, a, b)

$$I_{D,a,b}(\gamma) := I_{\mathbb{H},0,\infty}(\varphi(\gamma))$$

D a simply connected domain, a, b two boundary points (prime ends)

where φ is a conformal map $D \to \mathbb{H}$ such that $\varphi(a) = 0$, $\varphi(b) = \infty$.

Rmk: The choice of φ is not unique

$\tilde{\varphi} := \lambda \cdot \varphi$ also works

But if W is the driving function of $\varphi(\gamma)$

$\tilde{W}$ - - - - - $\lambda\varphi(\gamma)$

then $\tilde{W}_t = \lambda W_{\lambda^{-2}t}$

$$\frac{1}{2}\int_0^\infty \dot{\tilde{W}}_t^2 \, dt = \frac{1}{2}\int_0^\infty \left(\lambda^{-1}\dot{W}_{\lambda^{-2}t}\right)^2 dt$$

$$= \frac{1}{2}\int_0^\infty \dot{W}_s^2 \, ds$$

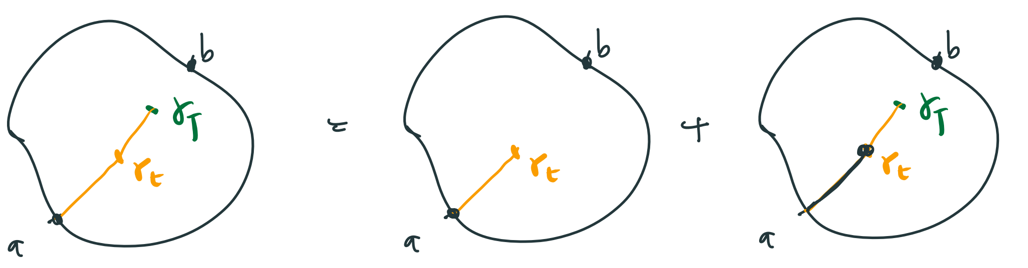
So $I_{D,a,b}(\gamma)$ is well defined, and does not depend on the choice of φ.

<u>Additivity</u>: for $0 < t < T$

$$\frac{1}{2}\int_0^T \dot{W}_S^2 \, ds = \frac{1}{2}\int_0^t \dot{W}_s^2 \, ds + \frac{1}{2}\int_t^T \dot{W}_s^2 \, ds$$

$$\frac{1}{2}\int_t^T \dot{W}_s^2 \, ds = \frac{1}{2}\int_0^{T-t} \left[\frac{d}{dr}\left(W_{t+r} - W_t\right)\right]^2 dr$$

$$I_{D,a,b}(\gamma[0,T]) = I_{D,a,b}(\gamma[0,t]) + I_{D \setminus \gamma[0,t],\, \gamma_t,\, b}(\gamma[t,T])$$



Conformal invariance

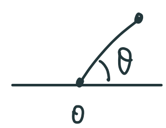
$$\varphi : (D,a,b) \to (\tilde{D},\tilde{a},\tilde{b})$$

Then $\quad I_{D,a,b}(\gamma) = I_{\tilde{D},\tilde{a},\tilde{b}}(\varphi(\gamma))$

Minimizer of $I_{D,a,b}$ is $\varphi^{-1}(i\mathbb{R}_+)$

namely the hyperbolic geodesic of (D,a,b).

or SLE_0

Example: • $I_{\mathbb{H},0,\infty}\Big($  $\Big)$ with $\theta \neq \frac{\pi}{2}$ is ∞.

In fact, $W_t = c(\theta)\sqrt{t}$

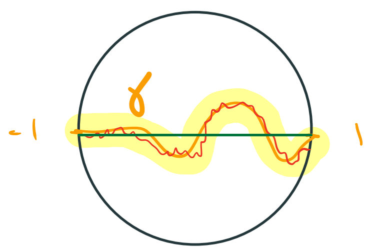
• 

$\Rightarrow I(\gamma) = \infty$

III) SLE_{0+} large deviations

Thm (Peltola W. 2020)

Let $\mathcal{X} := \{\text{simple curves in } \mathbb{D} \text{ from } -1 \text{ to } 1\}$ endowed with the Hausdorff distance. The random curve SLE_κ in $\mathcal{X}$ satisfy the LDP with good rate function $I_{\mathbb{D},-1,1}$ as $\kappa \to 0$.



$$\sim \exp\left(-\frac{I_{\mathbb{D},-1,1}(\gamma)}{\kappa}\right)$$

$$d_H(K, \tilde{K}) := \inf\{\varepsilon \geq 0 \mid \tilde{K} \subset \varepsilon\text{-neighbor of } K,\ K \subset \varepsilon\text{-neighbor of } \tilde{K}\}$$

compact sets of $\overline{\mathbb{D}}$

Remark: The map $W \longmapsto \overline{\bigcup_{t \geq 0} K_t}$ from $C_0(\mathbb{R}_+) \longmapsto \{\text{compact sets of } \overline{\mathbb{D}}\}$ is NOT continuous.

Proof: Schilder's theorem + some work for topology.

Remark: W has finite Dirichlet energy

$\Rightarrow$ W is Hölder-$\frac{1}{2}$ continuous

$$|W_t - W_s| = \left|\int_s^t \dot{W}_r \, dr\right| \le \sqrt{t-s}\sqrt{\int_s^t \dot{W}_r^2 \, dr}$$
$$\le \sqrt{t-s}\sqrt{2I(W)}$$

Brownian motion is only Hölder-$(\frac{1}{2}-\varepsilon)$ continuous

$\Rightarrow$ Finite energy W is more regular than any $\sqrt{\kappa}B$

Fact:

$\sqrt{\kappa}B \rightsquigarrow SLE_\kappa$ has Hausdorff dimension $1+\frac{\kappa}{8}$

Beffara

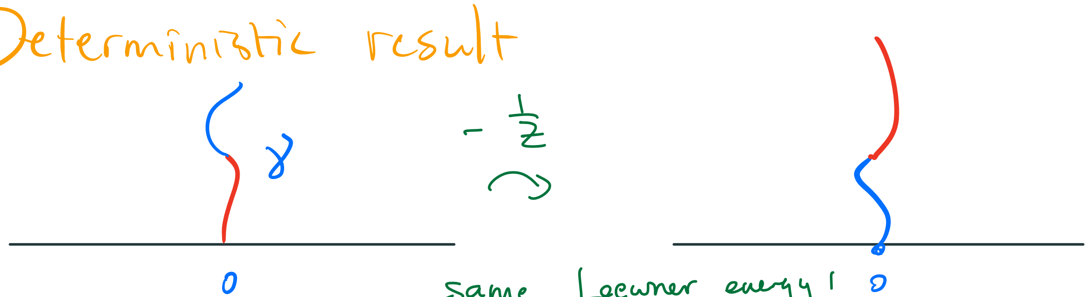
$W \rightsquigarrow \gamma$ with finite Loewner energy is rectifiable quasi-chord.

[Fritz: Shekhar]

Thm (W. '19) γ in (D, a, b)

$$I_{D,a,b}(\gamma) = I_{D,b,a}(\gamma^R)$$

Deterministic result



same Loewner energy!

Probabilistic Proof: SLE_κ is reversible

(for $0 \le \kappa \le 8$)

Zhan, Miller-Sheffield

$$\mathbb{P}(SLE_\kappa \text{ stays close to } \gamma \text{ in } (D, a, b))$$

Hausdorff

$$= \mathbb{P}(SLE_\kappa^R \text{ stays close to } \gamma^R \text{ in } (D, b, a))$$

$$= \mathbb{P}(SLE_\kappa \text{ stays close to } \gamma^R \text{ in } (D, b, a))$$

$$\sim \exp\left(\frac{-I_{D,a,b}(\gamma)}{\kappa}\right) \qquad \exp\left(\frac{-I_{D,b,a}(\gamma^R)}{\kappa}\right)$$