Large deviations of SLE and Weil-Petersson Teichmüller space

Lecture 2

— Yilin Wang (MIT)

Loewner energy

$SLE_\kappa \longleftrightarrow$ Weil-Petersson Teichmüller space

Schramm-Loewner evolutions

Probabilistic

- Stochastic analysis
- Conformally invariant 2D-statistical mechanics model
- Random planar maps
- 2D-quantum gravity
- Conformal field theory

Deterministic

- Quasicircles
- Quasiconformal mapping
- Geometric function theory
- Complex structures on Riemann surfaces
- Kähler geometry
- String theory

Yesterday:

I) Brownian Motion and Dirichlet energy

Schilder's theorem

II) SLE and Loewner energy

$W = \sqrt{\kappa} B$ $\qquad I(\gamma) = \frac{1}{2}\int_0^\infty \dot{W}^2(t)\, dt$

III) SLE_{0+} large deviations

Energy reversibility from SLE reversibility

Today:

I) Loop energy

II) Weil-Petersson Teichmüller space

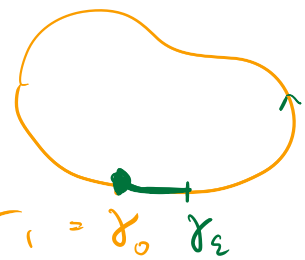
On Thursday:

I) Radial SLE_∞ large deviations

II) Foliations by Weil-Petersson quasicircles

I) Loewner energy for a Jordan curve

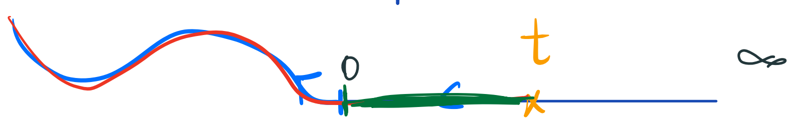
[Rohde, W.]



$\gamma : [0,1]/_{0\sim 1} \to \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

Def: $$\boxed{I^L(\gamma, \gamma_0) := \lim_{\varepsilon \to 0} I_{\hat{\mathbb{C}} \setminus \gamma_{[0,\varepsilon]}}(\gamma_{[\varepsilon, 1]})}$$

Example: $\gamma = \eta \cup \mathbb{R}_+$, $\gamma_0 = \infty$

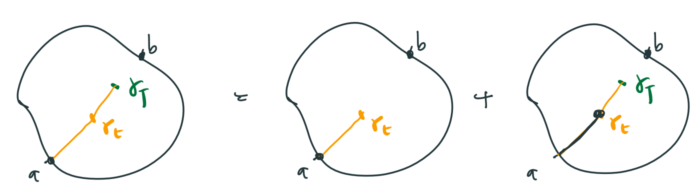


$$I^L(\gamma, \infty) = \lim_{t \to \infty} I_{\mathbb{C} \setminus [t,\infty)}(\eta \cup [0,t])$$

additivity
$$= \lim_{t \to \infty} \underbrace{I_{\mathbb{C} \setminus [t,\infty)}([0,t])}_{=0} + I_{\mathbb{C} \setminus \mathbb{R}_+}(\eta)$$

$$= I_{\mathbb{C} \setminus \mathbb{R}_+}(\eta)$$

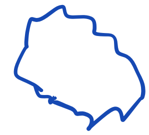
(Recall



)

$\Rightarrow$ Loop energy generalizes the chordal energy.

Q: Large deviations of SLE_{0+} loop?



Thm. [Rohde, W.] $I^L(\gamma, \cancel{\gamma_0})$ is independent of the root γ_0.

- $I^L(\gamma) = 0 \iff \gamma$ is a circle
- $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ conformal $\frac{az+b}{cz+d} \Rightarrow I^L(\varphi(\gamma)) = I^L(\gamma)$.
- $I^L(\gamma) < \infty \Rightarrow \gamma$ is a rectifiable quasicircle.

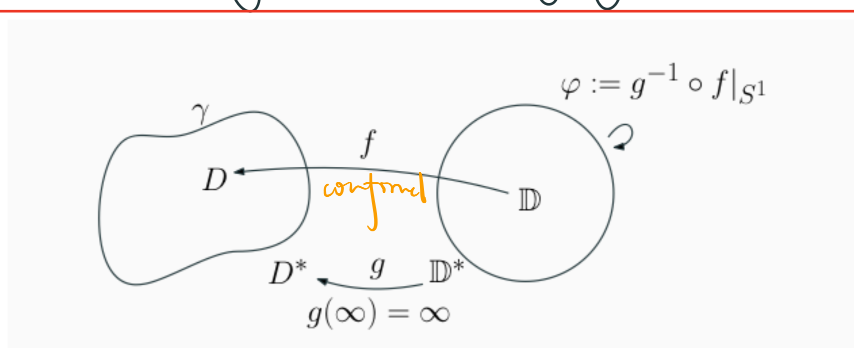
Thm. [W. '19 Invent. Math.]

$I^L(\gamma) < \infty$ iff γ is a Weil-Petersson quasicircle

Moreover,

$$\underbrace{\int_{\mathbb{D}} |\nabla \log |f'||^2 \, |dz|^2}_{\parallel}$$

$$\boxed{I^L(\gamma) = \frac{1}{\pi}\int_{\mathbb{D}} \left|\frac{f''}{f'}\right|^2 |dz|^2 + \frac{1}{\pi}\int_{\mathbb{D}^*} \left|\frac{g''}{g'}\right|^2 |dz|^2 + 4\log|f'(0)| - 4\log|g'(\infty)|}$$



• Simple expression! • Root-invariance

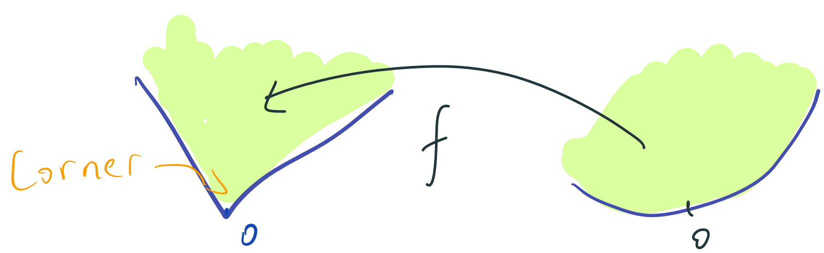
II) Weil-Petersson quasicircles

Def: A Weil-Petersson quasicircle is a Jordan curve γ such that

$$\int_{\mathbb{D}} \left|\frac{f''}{f'}\right|^2 |dz|^2 < \infty$$

Example: γ is $C^{2+\varepsilon}$ smooth $\Rightarrow$ WP

(In fact $C^{1.5+\varepsilon}$ suffices)



$$f(z) \sim z^{\alpha} \qquad \text{for} \quad \alpha \in (0,2)\setminus 1$$

$$\Rightarrow \frac{f''(z)}{f'(z)} \sim \frac{\alpha-1}{z} \quad \text{is not } L^2\text{-integrable}$$

A widely studied class of curves.

Motivated by string theory (smooth part)

Bowick - Rajeev '87 Witten '88 . Kirillov - Yurev '88

Hong, Nag, Pekonen . Sullivan '90 s

Studied from geometric/analytic perspectives

Cui '01, Takhtajan - Teo '06, Shen . Tang . Wu '13 - '20

Bishop '20

Application to conformal field theory

Radnell - Schippers - Staubach '17

Application to computer vision, KdV equations

Sharon - Mumford '06, Kushnarev '09

Schonbek - Todorov - Zubelli '99

Relation to holographic principles $\hat{\mathbb{C}} \leftrightarrow \mathbb{H}^3$

Bishop '20 (extending [Alexakis - Mazzeo] [Anderson])

Relation to random conformal geometry

W. 16-19 Viklund - W. '19 '20

WEIL-PETERSSON CURVES, β-NUMBERS, AND MINIMAL SURFACES

CHRISTOPHER J. BISHOP 2020

Definition	Description
1	$\log f'$ in Dirichlet class
2	Schwarzian derivative
3	QC dilatation in L^2
4	conformal welding
5	$\exp(i \log f')$ in $H^{1/2}$
6	arclength parameterization in $H^{3/2}$
7	tangents in $H^{1/2}$
8	finite Möbius energy
9	Jones conjecture
10	good polygonal approximations
11	β^2-sum is finite
12	Menger curvature
13	biLipschitz involutions
14	ε^2-sum is finite
15	δ-thickness in L^2
16	finite total curvature surface
17	minimal surface of finite curvature
18	additive isoperimetric bound
19	finite renormalized area
20	dyadic cylinder
21	closure of smooth curves in $T_0(1)$
22	P_φ^- is Hilbert-Schmidt
23	double hits by random lines
24	finite Loewner energy
25	large deviations of SLE(0^+)
26	Brownian loop measure

⋮

Definition

$\in$ Geometric function theory (involving conformal maps)

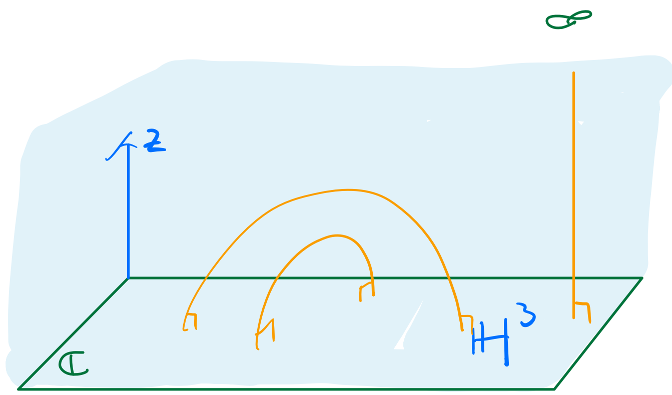
geometric characterization (only about curve, length, distance ..)

hyperbolic 3-space holography

Teichmüller Theory

SLE related

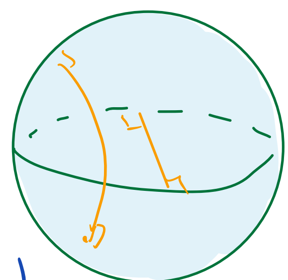
representation theoretic



$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}$$

$$\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$$

$\hat{\mathbb{C}}$



$$ds^2 = \frac{4\,\|dX\|^2}{(1 - \|X\|^2)^2}$$

$$\frac{az + b}{cz + d}$$

$$\mathbb{H}^3 = \{X = (x, y, z) \mid \|X\|^2 < 1\}$$

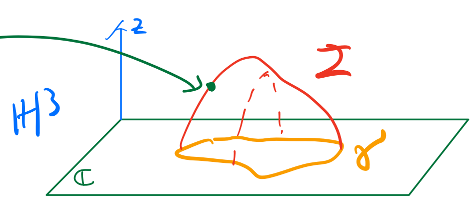
$PSL(2, \mathbb{C}) = \{$ Conformal automorphism of $\hat{\mathbb{C}}\}$
$= \{$ Isometry of $\mathbb{H}^3\}$

Thm (Bishop '20)

γ is WP iff γ bounds a minimal surface in $\mathbb{H}^3$ with total curvature $< \infty$.

not quantitative

principal curvature $k, -k$



$$\int_\Sigma k^2 \, dA_{hyp} < \infty$$

Loewner energy plays an important role to the geometry of this class:

{Weil-Petersson quasicircles} has an

∞-dimensional Kähler-Einstein manifold structure. and [Loewner energy] is a Kähler potential for the Kähler metric.

— By [Takhtajan-Teo '06]

Universal Liouville action

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Weil-Petersson Metric on the Universal Teichmüller Space

Leon A. Takhtajan

Lee-Peng Teo

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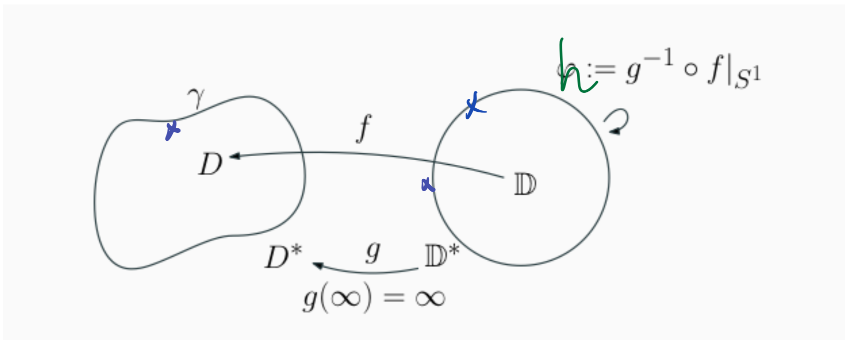
American Mathematical Society

More precisely

Associate γ to its welding homeomorphism $S^1 \to S^1$



[Beurling – Ahlfors]

γ is quasicircle $\Leftrightarrow$ $h \in QS(S^1)$

h is quasisymmetric

if $\exists M > 0$, $\forall t \in \mathbb{R}$, $\theta \neq 0$

$$\frac{1}{M} \leq \left| \frac{h(e^{i(t+\theta)}) - h(e^{it})}{h(e^{it}) - h(e^{i(t-\theta)})} \right| \leq M$$

Universal Teichmüller space

$T(1) = \text{Möb}(S^1) \backslash QS(S^1)$ quasicircles

$\{h \in QS(S^1) \mid h \text{ fixes } \pm 1, -i\}$

$\cup$

$T_0(1) = \text{Möb}(S^1) \backslash WP(S^1)$ Weil-Petersson quasicircles

$\cup$ Completion ([TT])

$\text{Möb}(S^1) \backslash \text{Diff}(S^1)$ C^∞ smooth curves

has a unique homogeneous kähler metric

Weil-Petersson metric

- $T_0(1)$ has a right-invariant Kähler structure

compatible (g, J, ω)

symplectic form

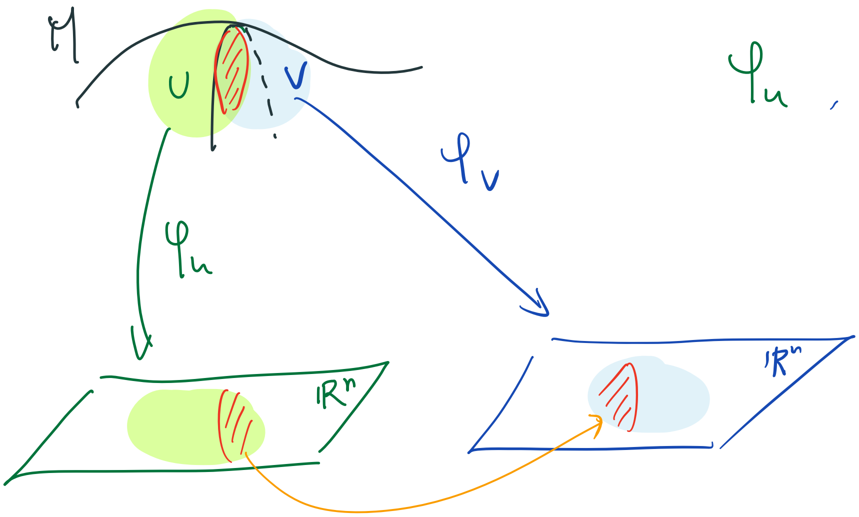
almost complex structure

Riemannian metric

Unique up to trivial scaling factor

A digression (finite dimensional Kähler structure)

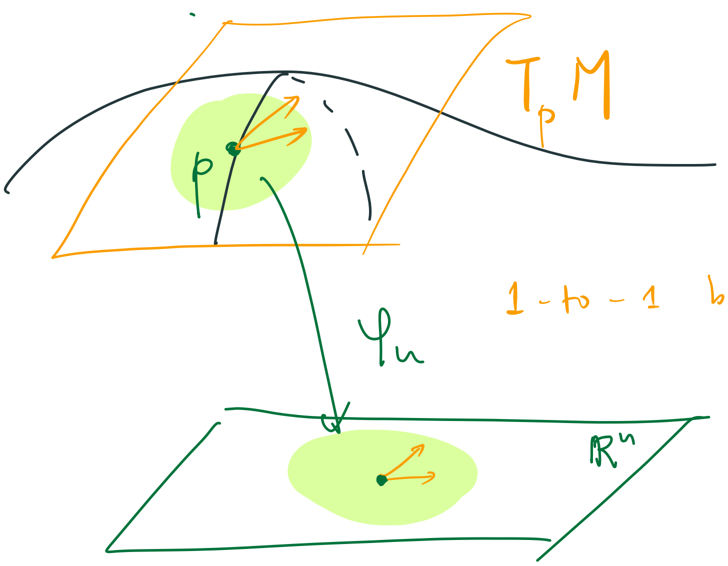
A manifold ($\underbrace{\varphi_U : U \to \varphi_U(U)}_{\text{homeomorphism}} \subset \mathbb{R}^n$)



φ_U, φ_V give local coordinates.

$\varphi_V \circ \varphi_U^{-1}$ C^∞-differentiable $\Rightarrow$ Smooth manifold

We define the tangent space



1-to-1 bijection

$$\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right)$$

a basis

- ## Riemannian metric

$$g_p = \sum_{i,j=1}^{n} g_{ij}(p)\, dx^i \otimes dx^j$$

where $(g_{ij})_{ij}$ is symmetric positive definite.

and g_{ij} is smooth in p.

- ## Complex structure (even dimension only)

(Replace $\mathbb{R}^{2n}$ by $\mathbb{C}^n$)

and transition maps $\varphi_v \circ \varphi_u^{-1}$ are biholomorphic (trickier for ∞-dim)

If $z_k = x_k + i y_k$

$$J : T_pM \to T_pM \quad \text{with } J^2 = -Id$$

$\mathbb{R}^{2n} \rightsquigarrow \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^n}\right)$

where $J\left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial y^k}$ $\quad J\left(\frac{\partial}{\partial y^k}\right) = -\frac{\partial}{\partial x^k}$

"J is the multiplication by i"

- Symplectic structure (even dimension)

$$\omega_p = \sum_{i<j} \omega_{ij}(p)\, dx_i \wedge dx_j$$

- Non degenerate

$$\omega_p(u,v) = 0 \quad \forall v \in T_pM \implies u = 0$$

- closed

$$\underset{\substack{\parallel \\ 0}}{d\omega} = \sum_{i<j} \sum_k \frac{\partial}{\partial x^k} \omega_{ij}\, dx_k \wedge dx_i \wedge dx_j$$

- g, J, ω compatible ⇔ Kähler

$$\omega(u, Jv) = g(u,v) \qquad \forall\, u, v \in T_pM, \ \forall p$$

- Kähler potential $I_U : U \to \mathbb{R}$

In general only defined locally

Such that $i\,\partial\bar{\partial} I_U = \omega$

complex notation
$$dz_k = dx_k + i\, dy_k$$
$$d\bar{z}_k = dx_k - i\, dy_k$$

$$\frac{\partial}{\partial z^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i \frac{\partial}{\partial y^k}\right)$$

$$\frac{\partial}{\partial \bar{z}^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i \frac{\partial}{\partial y^k}\right)$$

$$\partial = \sum_k \frac{\partial}{\partial z^k} dz_k$$ Dolbeault

$$\bar{\partial} = \sum_k \frac{\partial}{\partial \bar{z}^k} d\bar{z}_k$$

Namely $$\omega = \sum_{i,j} i \frac{\partial^2 I_u}{\partial z^i \, \partial \bar{z}^j} dz_i \wedge d\bar{z}_j$$

Example : $\mathbb{C} = \{ z = x+iy \}$ $\quad T_p \mathbb{C} \ \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\}$

- $g = (dx)^2 + (dy)^2$
- $J\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y} \qquad J\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x}$
- $\omega = dx \wedge dy$
- $dz = dx + i dy \ , \quad d\bar{z} = dx - i dy$

Compatible ?

$$\omega\left(\frac{\partial}{\partial x}, J\left(\frac{\partial}{\partial y}\right)\right) = \omega\left(\frac{\partial}{\partial x}, -\frac{\partial}{\partial x}\right) = 0$$

$$\omega\left(\frac{\partial}{\partial x}, J\left(\frac{\partial}{\partial x}\right)\right) = \omega\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 1$$

$$\omega\left(\frac{\partial}{\partial y}, J\left(\frac{\partial}{\partial y}\right)\right) = \omega\left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right)$$

$$= \omega\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 1$$

$\shortparallel$

$g(\cdot,\cdot)$

Yes Kähler !

Kähler potential (globally defined)

$$I(z) = \frac{|z|^2}{2} = \frac{x^2+y^2}{2} = \frac{z\bar{z}}{2}$$

Check :

$$i\partial\bar{\partial} I(z) = i\partial\left(\frac{z}{2}\, d\bar{z}\right)$$
$$= \frac{i}{2}\, dz \wedge d\bar{z}$$
$$= \frac{i}{2}\,(dx + i dy) \wedge (dx - i dy)$$
$$= \frac{i}{2}\,(-2i\, dx \wedge dy) = \omega$$

$\longrightarrow$ End of digression

Universal Teichmüller space

$T(1) = \text{Möb}(S^1) \backslash QS(S^1)$ quasicircles

$\cup$

$T_0(1) = \text{Möb}(S^1) \backslash WP(S^1)$ Weil-Petersson quasicircles

$\cup$ completion ([TT])

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has a unique homogeneous Kähler metric

Weil-Petersson metric

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compatible (g, J, ω)

ω: symplectic form

J: almost complex structure

g: Riemannian metric

Unique up to trivial scaling factor

(*)

Explicitly

On the tangent space of $[\mathrm{Id}_{S^1}]$

$$\left\{ u \in \mathrm{Vect}(S^1) \;\middle|\; u = \sum_{\substack{n \neq \pm 1 \\ n \neq 0}} u_n e^{in\theta} \frac{d}{d\theta}, \quad u_n = \overline{u_{-n}} \right\}$$

- $$J(u) = \sum_{n \neq \pm 1, 0} i \,\mathrm{sgn}(n)\, u_n e^{in\theta} \frac{d}{d\theta}.$$ (Hilbert transform)

$J^2 = -\mathrm{Id}$ $\quad J\left(\cos(n\theta)\frac{d}{d\theta}\right) = -\sin(n\theta)\frac{d}{d\theta}$

$J\left(\sin(n\theta)\frac{d}{d\theta}\right) = \cos(n\theta)\frac{d}{d\theta}$

Bowick - Rajeev '87

- $$\langle u, v \rangle_{W.P} = \mathrm{Re} \sum_{n=2}^{\infty} (n^3 - n)\, u_n v_{-n}$$

Unique! (up to multipli constant.)

- Compatible $\langle u, v \rangle_{W.P} = \omega(u, Jv)$

$\omega(u, v) = \langle u, -Jv \rangle_{W.P}$

Takhtajan - Teo defines the local charts on $T_0(1)$

$\Rightarrow J$ is integrable.

Thm. (Takhtajan - Teo '06)

$I^L([\varphi]) = I^L(\gamma)$ is a (global) Kähler potential of the Weil-Petersson Kähler form.

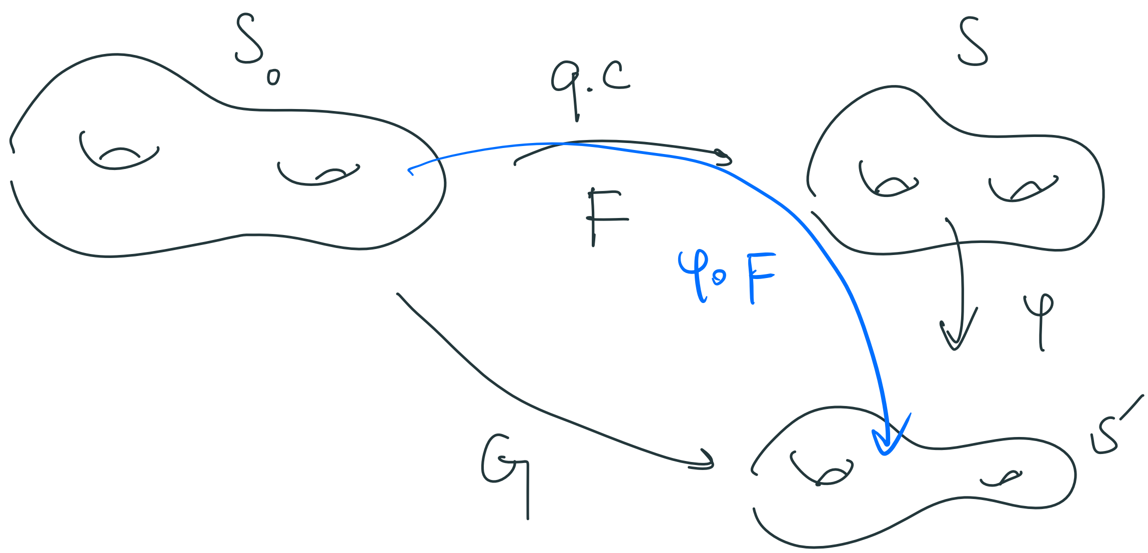
They show the result for the functional

$$\frac{1}{\pi}\int_{\mathbb{D}} \left|\frac{f''}{f'}\right|^2 |dz|^2 + \frac{1}{\pi}\int_{\mathbb{D}^*} \left|\frac{g''}{g'}\right|^2 |dz|^2 + 4\log|f'(0)| - 4\log|g'(\infty)|$$

Universal Liouville action

A digression on Teichmüller space

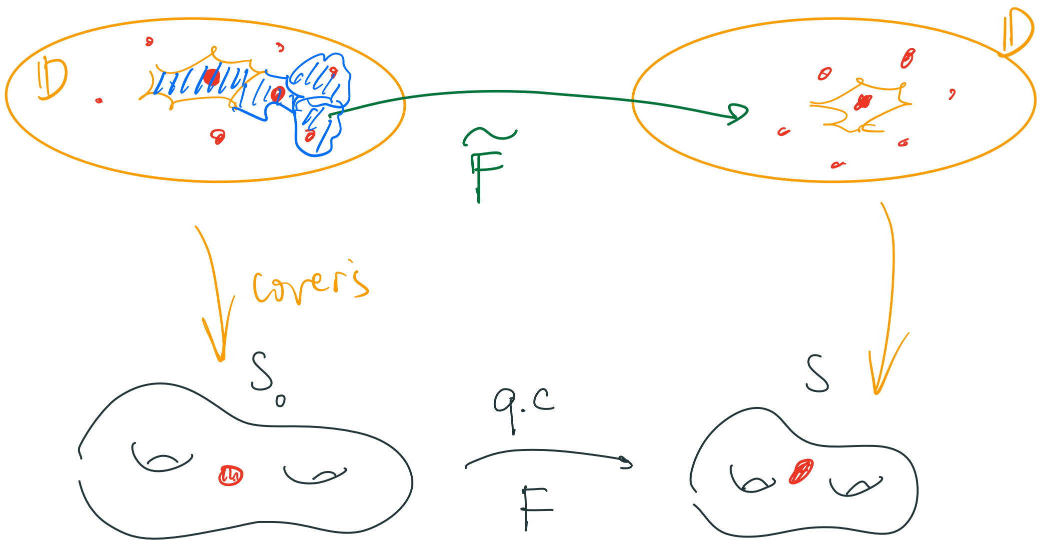
Teichmüller space of compact Riemann surfaces of genus g



$T_g = \{ (F : S_0 \to S) \} / \sim$

$\{F : S_0 \to S\} \underset{T_g}{\sim} \{G : S_0 \to S'\}$ iff $\exists\ \varphi : S \to S'$ conformal s.t $\varphi \circ F$ is homotopic to G.

When $g \geq 2$



$S_0 = \mathbb{D}/\Gamma_0$ $\qquad$ $S = \mathbb{D}/\Gamma$

where Γ_0 is a discrete subgroup of $\text{Möb}(\mathbb{D})$

Γ - - — — —

F lifts to a quasiconformal map of $\mathbb{D}$. $\tilde{F}$

$\tilde{F}: \mathbb{D} \to \mathbb{D}$ has boundary value h_F

$\Uparrow$
$QS(S^1)$

Thm: $\{F : S_0 \to S\} \underset{T_g}{\sim} \{G : S_0 \to S'\}$

iff $h_F \sim h_G$ in $T(1)$

Upshot:

$$T_g \hookrightarrow T(1)$$

Hence the name "universal"

End of digression.