Multi-dimensional Brownian Motion with Darning

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Outline

- Motivation: What is a “darning" process?
- Examples of darning processes
- Main results: heat kernel estimates of multi-dimensional Brownian motion with darnin

Question

What is a “darning" process?

Answer

We either

- “patch" the boundaries of multiple processes together, or
- “collapse" some part of the state space of a process to a singleton.

Examples and pictures of darning processes will be given soon.

Question

How to construct a darning process?

Answer

Very roughly speaking, in terms of Dirichlet forms.

Examples of darning processes

Example (Circular Brownion motion)

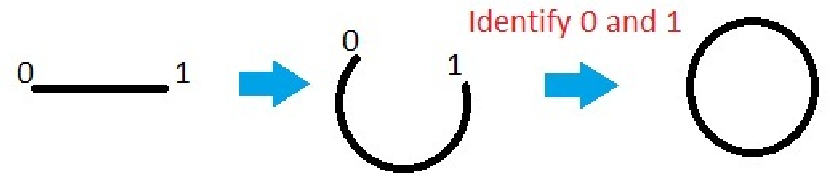
Idea:“Gluing" the two endpoints of an absorbing Brownian motion on an interval $I := (0, 1)$.

The Dirichlet form of circular Brownian motion is

$$\begin{cases} \mathcal{F} = \{u : u \in BL(I), u(0+) = u(1-)\} \cap L^2(I), \\ \mathcal{E}(u, v) = \dfrac{1}{2} \displaystyle\int_I u'(x)v'(x)dx, \end{cases}$$

where $BL(I) = \{u : u \text{ is absolutely continuous on } I \text{ with } \int_I (u')^2 dx < \infty\}$.

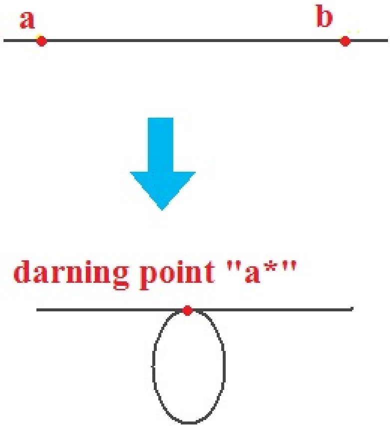
Picture of circular Brownian motion



Examples of darning processes

Example (Brownian motion with a “knot")

Idea: Identifying two points on $\mathbb{R}$ as a singleton.



(Continued)

Suppose we identify two points a and b on $\mathbb{R}$. The Dirichlet form of such a process is

$$\begin{cases} \mathcal{F} = \{u : u \in W^{1,2}(\mathbb{R}), u(a) = u(b)\}, \\ \mathcal{E}(u,v) = \frac{1}{2}\int_{\mathbb{R}} u'(x)v'(x)dx. \end{cases}$$

Remark

In the same way we may identify up to countably many non-accumulating points on $\mathbb{R}$ so that the “knot" has multiple loops.

Question

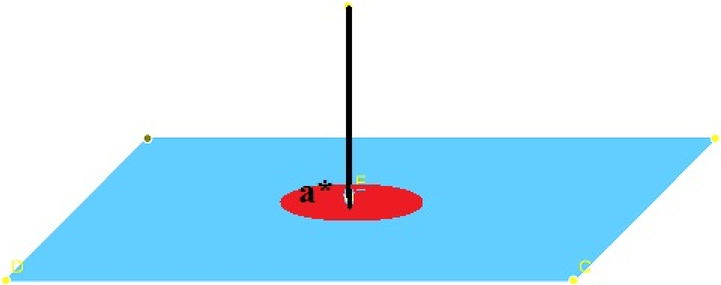
Why do we have to define a multi-dimensional Brownian motion as a “darning process"?

Answer

Even for the simplest case of $\mathbb{R}^2 \sqcup \mathbb{R}$, a standard 2-dimensional Brownian motion never hits a singleton!

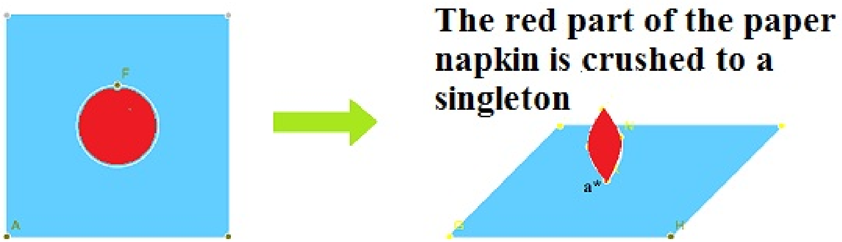
Solution

By “collapsing" the closure of an open set to a single point, one gets a “2-dimensional Brownian Motion" that does hit a singleton, which is called a *Brownian motion with darning*.



The red area is collapsed to a singleton

Another picture of this process



Question

What is heat kernel estimate?

Answer

A Markov process has a transition semigroup: $\mathbb{P}^x(X_t \in \cdot) := P_t(x, \cdot)$, which is a probability measure for every fixed pair of (t, x). In many cases, it is very hard to give the explicit expression of $P_t(x, \cdot)$.
We usually denote the density of $P_t(x, \cdot)$ by $p(t, x, y)$. Our goal is to find a function $f(t, x, y)$ such that there exists some constant $C > 0$ such that

$$\frac{1}{C} \cdot f(t, x, y) \asymp p(t, x, y) \asymp C \cdot f(t, x, y), \quad \text{for all } t, x, y.$$

Proposition (Small time heat kernel estimate)

There exist constants C_1, $C_2 > 0$, *such that*

$$p(t,x,y) \asymp \begin{cases} \frac{1}{\sqrt{t}} e^{-\frac{C_1|x-y|^2}{t}}, & |y|_g \le 1; \\ \frac{1}{t} e^{-\frac{C_2|x-y|^2}{t}}, & |y|_g > 1, \end{cases}$$

for all $x \in \mathbb{R}$, $y \in \mathbb{R}^2 \backslash B_\epsilon$, $t \in [0,1]$.

Theorem (Small time heat kernel estimate)

There exist constants $C_i > 0$, $3 \le i \le 5$, *such that for all* $t \in [0,1]$, $x, y \in \mathbb{R}^2 \backslash B_\epsilon$,

$$p(t,x,y) \asymp \begin{cases} \frac{1}{\sqrt{t}} e^{-\frac{C_3 |x-y|_g^2}{t}} + \frac{1}{t}\left(1 \wedge \frac{|x|_g}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|_g}{\sqrt{t}}\right) \quad e^{-\frac{C_4 |x-y|_e^2}{t}}, & |x|_g < 1,\ |y|_g < 1; \\ \frac{1}{t} e^{-\frac{C_5 |x-y|_g^2}{t}}, & \text{otherwise.} \end{cases}$$

Theorem (Large time heat kernel estimate)

There exist constants $C_6 > 0$ such that

$$p^{(X)}(t,x,y) \asymp \frac{1}{\sqrt{t}} e^{-\frac{C_6|x-y|^2}{t}}, \qquad t \in [0,\infty), \quad x,y \in \mathbb{R}.$$

Theorem (Large time heat kernel estimate)

There exists constant $C_7 > 0$ such that

$$p^{(X)}(t,x,y) \asymp \frac{1}{\sqrt{t}} e^{-\frac{C_7 |x-y|_g^2}{t}}, \qquad t > 1, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}^2.$$

Theorem (Large time heat kernel estimate)

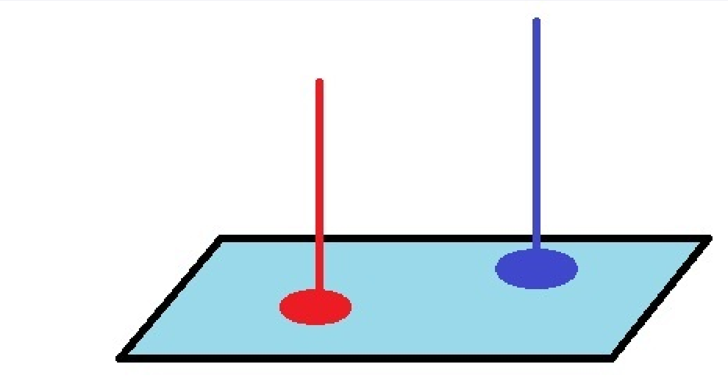
There exist constants $C_i > 0$, $8 \le i \le 10$, such that for all $t > 1$, $x, y \in \mathbb{R}^2 \backslash B_\epsilon$,

$$p^{(X)}(t,x,y) \asymp \begin{cases} \frac{1}{t} e^{-\frac{C_8 |x-y|_g^2}{t}} + \frac{1}{\sqrt{t}} e^{-\frac{C_9 |x|_g^2 + |y|_g^2}{t}}, & |x|_g > \sqrt{t},\ |y|_g > \sqrt{t}; \\ \frac{1}{\sqrt{t}} e^{-\frac{C_{10} |x-y|_g^2}{t}}, & \text{otherwise.} \end{cases}$$

Other obtained results related to small time HKE

- Hölder continuity of parabolic functions
- Counter example to parabolic Harnack inequality
- Case of multiple straight lines
- Case of a “handle" attached to a plane

Picture of the case of multiple straight lines



Picture of a “handle" attached to a plane