RANDOM MAPS

G. MIERMONT (UNIV. PARIS-SUD & CNRS/PIMS)

Abstract. Course during the PIMS/Mprime probability summer school at UBC. Notes by L. Silberman who is responsible for all errors and omissions (corrections would be appreciated – send to lior@math.ubc.ca)

Contents

Date: June 2012.

Overview and motivation

Definition 1. A *plane map* is an embedding of a finite connected (multi)graph in S^2, considered up to the action of $\mathrm{Homeo}^+(S^2)$.

For a map $\mathbf{m}$ let $V(\mathbf{m})$ be the vertices, $E(\mathbf{m})$ be the edges, $F(\mathbf{m})$ be the faces.

Fact 2. *(Euler)* $\#V(\mathbf{m}) - \#E(\mathbf{m}) + \#F(\mathbf{m}) = 2$

Definition 3. A *rooted map* is a map, together with a choice of an oriented edge.

Motivation.

- Study *maps* as *discretizations* of 2d Riemannian manifolds.
  - Consider each face of $\mathbf{m}$ as a flat regular polygon where edge lengths are constant.
- This comes from 2d *quantum gravity* in which a basic object is the path integral
$$\int_{\mathcal{R}(M)/\,\mathrm{Diff}^+(M)} [\mathcal{D}g] \exp\left(-S(g)\right)$$
  - Here M is a 2d orientable manifold, $\mathcal{R}(M)$ is the space of Riemannian metrics on M, Diff^+ is the group of orientation-preserving diffeomorphisms, $S(g) = \alpha A_g(M) + \beta\chi(M)$ is the action.
  - $[\mathcal{D}g]$ is the "volume measure", which is not formally defined.
- How to deal with this measure?
  - Quantum Liouville Theory (Polyakov, David ...): Write $g = e^{2u}h^* g_0(\mu)$ where u is the conformal factor and$g_0(\mu)$ is the hyperbolic metric, parametrized by the *moduli* μ, and $h \in \mathrm{Diff}^+(M)$. Computing a formal "Jacobian" for this transformation gives the following partition function:
$$\int \exp\left(-S_L(u, g_0(\mu))\right) \mathcal{D}u$$
    where $S_L$is the *Liouville action* (a quadratic form in u) and for the measure on u can take it to be the *Gaussian free field.* This has been formalized by Duplantier–Sheffield.
  - Discretization: Replace
$$\int_{\mathcal{R}(M)/\,\mathrm{Diff}^+(M)} [\mathcal{D}g] \to \sum_{T \in \mathrm{Tr}(M)} \delta_T$$
    where $\mathrm{Tr}(M)$ is the set of triangulations of M. Then one takes the *scaling limit.*
  - This is useful in the setting of 1d path integrals, where random walks approximate Brownian motion, who law is a Gaussian measure on paths.
- At the moment the links between the two approaches are not well-understood.

Step 0: Enumeration of maps.

Indirect methods. Tutte used generating function methods to count maps. For example:

$$\#\{\text{Rooted plane maps with } n \text{ edges}\} = \frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}.$$

- *Matrix integrals* provide a new method for obtaining Tutte's generating function equations, generalizing to maps of all genera and with coloured edges.

Example 4. We have

$$\frac{1}{N^2}\log\left(\frac{1}{Z_N}\int_{\mathcal{H}_N}\exp\left\{-\frac{t}{N}\operatorname{Tr}\left(\frac{H^4}{4}\right)-\operatorname{Tr}\left(\frac{H^2}{2}\right)\right\}\mathrm{d}H\right)=\sum_{g\geq 0}\frac{1}{N^{2g}}\sum_{\mathbf{q}\in\mathbf{Q}^{(g)}}\frac{(-t)^{\#V(\mathbf{q})}}{\operatorname{Aut}(\mathbf{q})}$$

where $\mathbf{Q}^{(g)}$ is the set of quadrangulations of genus q and $\mathcal{H}_N$ is the space of $N\times N$ Hermitian matrices equipped with Lebesgue measure $\mathrm{d}H$.

Bijective methods. The methods above do not generally apply well to the *metric aspects* of the maps (notable exception: the [non-rigorous] computation by Ambjørn–Watabiki (1995) of the 2-point function of triangulations). However, an approach by bijections with decorated trees was initiated by Cori–Vanquelin (1981), Arqès (1986) and fully developed by Schaeffer (1998-).

[Image of a random quadrangulation by J.-F. Marckert; noted that it looks like a topological sphere but metrically it is fractal and very different from the round sphere]

Scaling limits.

- Choose a rooted plane quadrangulation $Q_n \in \mathbf{Q}_n$ (n faces) uniformly at random.
- Endow $V(Q_n)$ with the graph distance d_{Q_n}.
- Typically $d_{Q_n}(u,v)\approx n^{1/4}$.

Theorem 5. *(Le Gall 2011, Miermont 2011) We have (convergence in distribution, in the sense of Gromov–Hausdorff topology)*

$$\left(V(Q_n),(8n/9)^{-1/4}d_{Q_n}\right)\xrightarrow[n\to\infty]{(d)}(S,D)$$

where (S,D) is the Brownian map*.*

Most of the course is devoted to the proof of this result. Other topics:

- Topology of the limit is that of S^2
- The 3-point function in the limiting space
- Replace quadrangulations with other models of random maps

0. Basic properties of maps

Basic terminology — maps.

- Surfaces will always be orientable, compact and connected. Write S_g for the surface of genus g ($S_0\approx S^2$, $S_1=\mathbb{T}^2$, ...),
- Graphs will be finite undirected multigraphs without half-edges, formally thought of as triples (V,E,ι) where V,E are finite sets (vertices, and edges) and $\iota\subset E\times V$ is the *incidence relation*.

Fix a surface S. An *oriented edge* on S is a continuous map $e\colon [0,1] \to S$ where either e is injective or $e \restriction_{[0,1)}$ is injective and $e(1) = e(0)$, considered up to reparametrization $e \mapsto e \circ h$ where $h \in \mathrm{Homeo}^+([0,1])$. An *edge* on S is a pair $\{e, \bar{e}\}$, where $\bar{e}$ is the edge $\bar{e}(t) = e(1-t)$. For an oriented edge e write $e^- = e(0)$, $e^+ = e(1)$.

Definition 6. An *embedded graph* in S is a graph $G = (V, E, \iota)$ where $V \subset S$ is finite, E is a set of edges in S (as defined above), the incidence relation is that $v \in V$ is an endpoint of $e \in E$ (we suppose that $e(0), e(1) \in V$ for all $e \in \vec{E}$), and such that for any two distinct non-opposite oriented edges $e, e' \in \vec{E}$ we have $e\left((0,1)\right) \cap e'\left((0,1)\right) = \emptyset$ and $e\left((0,1)\right) \cap V = \emptyset$. We will write $\vec{E} = \cup E$ for the set of oriented edges of G.

The *support* of G is $\mathrm{supp}(G) = V \cup \bigcup_{e \in \vec{E}} e\left([0,1]\right)$.

Definition 7. A *map* is an embedded graph $\mathbf{m}$ such that the connected components of $S \setminus \mathrm{supp}(\mathbf{m})$ are homeomorphic to discs.

Lemma 8. *A map is always connected as a graph.*

Definition 9. A *plane map* is a map on S^2.

Lemma 10. *Every connected embedded graph on S^2 is a map.*

Remark 11. This is not true for other surfaces. For example, $\mathbb{T}^2 \setminus \{\mathrm{pt}\}$ is not homeomorphic to a disc (it is homotopic to a circle).

Definition 12. A *rooted map* is a pair $(\mathbf{m}, e_*)$ where $\mathbf{m}$ is a map and $e_* \in \vec{E}$. Call e_*^- the *root vertex* of $\mathbf{m}$.

Definition 13. Let $\mathbf{m}$ on S, $\mathbf{m}'$ on S' be two (rooted) maps. We say that they are *isomorphic* if there is a homeomorphism $\varphi\colon S \to S'$ which induces a graph isomorphism: u, e are incident iff $h(u), h \circ e$ are incident (and $h \circ e_* = e'_*$).

From now on we will usually identify isomorphic maps. Of course one works with specific maps, but one cares about isomorphism classes.

Fact 14. *An automorphism of a rooted map must fix the oriented edges.*

Proof. Amounts to showing that a map is given by the cyclic order of the edges at each vertex. □

Vertices, Faces and degrees. Fix a map $\mathbf{m}$ on a surface S.

Recall that the faces $F(\mathbf{m})$ are the connected components of $S \setminus \mathrm{supp}(\mathbf{m})$. If $e \in \vec{E}(\mathbf{m}$ is an oriented edge include it in the face that is to its left which we denote f_e (here we use the orientability of S). Note that it is possible that $f_e = f_{\bar{e}}$.

Definition 15. Let $v \in V(\mathbf{m})$. Its degree *is* $\deg(v)$ *is* $\#\left\{e \in \vec{E}(\mathbf{m} \mid e^- = v\right\}$. *Let* $f \in F(\mathbf{m})$. *Its* degree *is* $\deg(f) = \#\left\{e \in \vec{E}(\mathbf{m} \mid f_e = f\right\}$

Remark 16 (The "fat graph" representation). Up to isomorphism a map is determined by the combinatorial data of the degrees of its faces and the incidence relation of the faces (which edge of each face is glued to which face). It follows that the set of isomorphism classes of maps is countable.

Euler's formula.

Theorem 17 (Euler)**.** *Let* $\mathbf{m}$ *be a map on a surface* S_g*. Then* $\#V(\mathbf{m}) - \#E(\mathbf{m}) + \#F(\mathbf{m}) = 2 - 2g \overset{def}{=} \chi(S_g)$*.*

Example 18. Let $\mathbf{m}$ be an embedding of K_5 on a surface of genus g. Then every face has degree at least 3, since there are no self-loops (faces of degree 1) and parallel edges (faces of degree 2). Thus

$$2\#E(\mathbf{m}) = \sum_{f \in F(\mathbf{m})} \deg(f) \geq 3\#F(\mathbf{m})$$

so $\#F(\mathbf{m}) \leq \frac{20}{3}$. Also, by Euler's formula $\#F(\mathbf{m}) = 2 - 2g + 10 - 5 = 7 - 2g$. It follows that $2g \geq 7 - \frac{20}{3} > 0$ so $g > 0$ and we find that $K_5$is not planar.

Exercise 19. Show that $K_{3,3}$ is also non-planar.

Lemma 20. *A plane map is a tree (as an abstract graph) if and only if it has one face).*

Proof. By the Jordan curve theorem a simple cycle in the underlying graph will divide the sphere into two discs; the graph complement will have connected components on each side hence at least two faces. Conversely, if the underlying graph is a tree then the complement is connected (trace a polygon around the tree), and this actually shows that every tree embedded in the sphere is a map. □

Exercise 21. Prove Euler's formula for $g = 0$ by inductively removing edges until one is left with a *tree* (a plane map with one face).

Part 1. **Random trees**

1. The contour process and its scaling limit

1.1. **Random trees.** Let $\mathbf{T}$ be the set of rooted plane trees, $\mathbf{T}_n \subset \mathbf{T}$ the set of trees with n edges.

Theorem 22. $\#\mathbf{T}_n = C(n) = \frac{1}{n+1}\binom{2n}{n}$.

Combinatorial Proof. Let $A(z) = \sum_{n \geq 0} \#\mathbf{T}_n z^n$. Then $A(z) = 1 + z(A(z))^2$ since every rooted tree can be written as the union its root and the two trees obtained by removing the root (oriented by the next edge at each vertex, respectively. □

Geometric Proof. Given a rooted tree $\mathbf{t}$ walk around it, starting at the root. Let $C_{\mathbf{t}}(k)$ be the distance from the root at time k (so $C(0) = C(2n) = 0$, and add $C(2n+1) = -1$). Make this into the graph of a function by interpolating linearly. Conversely, given any sequence ("lattice path") $C(0), C(0), \cdots, C(2n) \in \mathbb{Z}_{\geq 0}$ with $|C(i+1) - C_i| = 1$ we can define a tree by interpolating linearly in $[0, 2n]$ and identifying times s_1, s_2 if $C(s_1) = C(s_2)$ and the interval $[s_1, s_2] \times \{C(s_2)\}$ lies entirely below the graph of $C(s)$. Then the map $\mathbf{t} \to C_{\mathbf{t}}$ gives a bijection between trees and walks. □

Note 23. The function $C(n)$ will be called the *contour function* of the tree.

Generating random trees. Start a random walk S_t on $\mathbb{Z}$ where $S_0 = 0$. Let τ_i be the stopping time $\min\{t \mid S_t = -i\}$, so that $0 = \tau_0 < \tau_1 < \cdots$. Then the "paths" $(S_{\tau_i+k} - S_{\tau_i} = i + S_{\tau_i+k})_{k=0}^{\tau_{i+1}-\tau_i}$ form an iid sequence of excursions. To the ith path associate the tree $T^{(i)}$

Lemma 24. *The degree of the root of $T^{(0)}$ has distribution $\mathrm{Geom}\left(\frac{1}{2}\right)$, and the subtrees at the children of the root are of same distribution as $T^{(0)}$.*

Proof. The neighbours of the root consist of the times $t \leq \tau_1$ such that $S_t = 0$ (by construction they are all visible from the root). Moreover, at every time where $S_t = 0$ we end the excursion with probability $\frac{1}{2}$. Finally, between consecutive times where $S_{t_1} = S_{t_2} = 0$, S is precisely a bridge starting at 1 at time S_{t_1+1} and getting to 0 for the first time at time t_2. □

Corollary 25.

(1) *$T^{(0)}$ is a Galton–Watson tree with child distribution* $\mathrm{Geom}\left(\frac{1}{2}\right)$

(2) *We have*

$$\mathbb{P}\left(T^{(0)} = \mathbf{t}\right) = \prod_{v \in V(\mathbf{t})} \frac{1}{2^{\deg_o(v)+1}} = 2^{-(\#V + \#E)} = 2^{-(1+2\#E)}$$

where $\deg_o(v)$ *is the* out-degree *of v, so that $\sum_v \deg_o(v) = \#E$. We also used that $\#V = \#E + 1$.*

(3) *Conditioned on $\#E(T^{(0)}) = n$, $T^{(0)}$ is uniform on the set of trees with n edges.*

1.2. **The Brownian bridge.**

Theorem 26 (Donkers)**.** *For $t \in \mathbb{R}_{\geq 0}$ set $S_t = (1 - \{t\})S_{\lfloor t \rfloor} + \{t\}\, S_{\lceil t \rceil}$. Then*

$$\left(t \mapsto \frac{S_{2nt}}{\sqrt{2nt}}\right) \xrightarrow[n\to\infty]{(d)} (t \mapsto B_t)$$

where B_t is a standard Brownian motion, and the convergence in $C(\mathbb{R}_{\geq 0}, \mathbb{R})$ is uniform on compact sets.

As above, for $x \geq 0$ set $\kappa_x = \inf\{t \mid B_t < -x\}$, $\kappa_{x-} = \lim_{y \to x} \kappa_y$. We then set $e_t^{(x)} = x + B_{\kappa_{x-}}$ for $0 \leq t \leq \kappa_x - \kappa_{x-}$.

Remark 27. $\{x \mid \kappa_x > \kappa_{x-}\}$ is countable a.s.

The functions $e^{(x)}$ belong to $\mathcal{E} = \cup_{\zeta > 0} C([0, \zeta], \mathbb{R})$, and for $f \in \mathcal{E}$ let $\zeta(f)$ be the duration of f. With the metric $\mathrm{d}(f, g) = \sup_{t \geq 0} |f(t \wedge \zeta(f)) - g(t \wedge \zeta(g))| + |\zeta(f) - \zeta(g)|$ this is a Polish space. Enumerating $\{x_i\}_{i \in I}$ the set of x such that $\kappa_x > \kappa_{x-}$, let $\mathcal{N} = \sum_{i \in I} \delta_{(x_i, e^{(x_i)})}$ as a measure on the product $\mathbb{R} \times \mathcal{E}$.

Theorem 28 (Itô)**.** *The measure $\mathcal{N}$ is a Poisson random measure on $\mathbb{R}_{>0} \times \mathcal{E}$ with the intensity measure $2\,\mathrm{d}x \otimes n(\mathrm{d}e)$ where n is a σ-finite measure on $\mathcal{E}$ ("Itô measure of BM").*

1.3. **Scaling limit.**

We can now calculate the scaling limit of $C_{T^{(0)}}$ conditioned on having at least n vertices:

Corollary 29. *Let* $R_1 = \inf\left\{x \geq 0 \mid \zeta(e^{(x)}) \geq 1\right\}$*, so that* $e^{(R_1)}$ *is the first arc of* B_t *with length*≥ 1*. Then, conditioned on* $\#E(T^{(0)}) \geq n$*,*

$$\left(\frac{C_{T^{(0)}}(2n\cdot)}{\sqrt{2n}}\right) \restriction_{\left[0, \frac{2\#E(T^{(0)}+1}{2n}\right]} \xrightarrow{(d)} e^{(R_1)} .$$

Moreover, the law of $e^{(R_1)}$ *is* $n\left(\cdot \mid \zeta(f) \geq 1\right)$ *(here* $n\left(E|A\right) = \frac{n(E\cap A)}{n(A)}$ *which makes sense if* $0 < n(A) < \infty$*.*

Proof. It is enough to show that $\frac{C_{T^{(r_1)}}(2ns)}{\sqrt{2n}}$ converges to the above limit, where $r_1 = \inf\left\{k \mid \#E(T^{(k)}) \geq n\right\}$. Now if the operation of taking the first excursion of duration≥ 1 was continuous in the uniform topology, we'd be done by the weak-* convergence of the SRW to Brownian motion. However, for BM local minima are a.s. attained only once, and for x large, $\kappa_x - \kappa_{x-} \neq 1$. □

Properties of n. Let $p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ be the heat kernel, $p_t(x,y) = p_t(y-x)$, $p_t^+(x,y) = p_t(x,y) - p_t(x,-y)$, $q_t(x) = \frac{x}{t} p_t(x)$. Then q_t is the probability density function for κ_x, for which we use the notation $\mathbb{P}\left(\kappa_x \in \mathrm{d}t\right) = q_t(x)\,\mathrm{d}t$.

Theorem 30.

(1) *For every measurable* $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ *,* $n\left(f(e_t)\mathbb{1}_{\{\zeta(f)>t\}}\right) = \int_0^\infty \mathrm{d}x q_t(x) f(x)$*.*
(2) *(Markov property) of* n*.*

Corollary 31. *For all* $0 < t_1 < \cdots < t_k$*,* $n\left(e_{t_1} \in \mathrm{d}x_1 \wedge \cdots \wedge e_{t_k} \in \mathrm{d}x_k \wedge \zeta(x) > t_k\right) = q_{t_1}(x_1)\,\mathrm{d}x_1 \cdot p^+_{t_2-t_1}(x_1,x_2)\,\mathrm{d}x_2 \cdots p^+_{t_k-t_{k-1}}(x_{k-1},x_k)\,\mathrm{d}x_k$*.*

Remark 32. It is possible to disintegrate n so that $n = \int_0^\infty n^{(t)} \mathbb{P}\left(\zeta(f) \in \mathrm{d}t\right)$ (and the latter marginal is known). Then $n^{(1)}$ is called the law of the normalized Brownian excursion $\mathbf{e}$. For example, scaling the time of $e^{(R_1)}$ by R_1 gives a random function with law $\mathbf{e}$.

Proposition 33. *For every* $t \in (0,1)$*,* F *measurable and non-negative on* $C([0,t])$*,*

$$n^{(t)}\left(F\left((s \mapsto e(s))_{0\leq s\leq t}\right)\right) = n\left(F\left((s \mapsto e(s))_{0\leq s\leq t}\right) \cdot \mathbb{1}_{\zeta(e)>t} \cdot 2\sqrt{2\pi} q_{1-t}(e(t))\right) .$$

In other words, $n^{(t)}$ *restricted to the* σ*-algebra* $\mathcal{F}_t$ *generated by* e_s *for* $s \leq t$ *is absolutely continuous with respect to* $n \restriction_{\mathcal{F}_t} \mathbb{1}_{\{\zeta(e)>t\}}$ *with density* $2\sqrt{2\pi} q_{1-t}(e(t))$*.*

Exercise 34. Show that if U is uniform in $[0,1]$ independent of $\mathbf{e}$ (with law $n^{(1)}$) then $\mathbb{P}\left(2\mathbf{e}_U \in \mathrm{d}x\right) = x e^{-\frac{x^2}{2}}\,\mathrm{d}x$.

1.4. Main Theorem: the scaling limit of the contour process.

Theorem 35. *If* T_n *is a uniform plane tree with* n *edges, and* C_n *is the contour process of* T_n *then*

$$\left(s \mapsto \frac{C_n(2ns)}{\sqrt{2n}}\right)_{0\leq s\leq 1} \xrightarrow[n\to\infty]{(d)} \mathbf{e}$$

in the space $C\left([0,1],\mathbb{R}\right)$ *with* $\|\cdot\|_\infty$*.*

Proof. One approach is to show that the finite marginals converge and then show tightness of the sequence.

Alternatively, let F be cts and bounded and let $0 < t < 1$, and consider

$$\begin{aligned}\mathbb{E}\left[F\left(s\mapsto\frac{C_n(2ns)}{\sqrt{2n}}\right)_{0\le s\le t}\right] &= \mathbb{E}\left[F\left(s\mapsto\frac{S_{2ns}}{\sqrt{2n}}\right)_{0\le s\le t}\Big|\tau_1=2n+1\right]\\ &= \frac{\mathbb{E}\left[F\left(s\mapsto\frac{S_{2ns}}{\sqrt{2n}}\right)_{0\le s\le t}\wedge\tau_1=2n+1\right]}{\mathbb{P}\left(\tau_1=2n+1\right)}\\ &= \frac{\mathbb{E}\left[F\left(s\mapsto\frac{S_{2ns}}{\sqrt{2n}}\right)_{0\le s\le t}\mathbb{1}_{\{\tau_1>\lceil 2nt\rceil\}}\mathbb{P}_{S_{\lceil 2nt\rceil}}\left(\tau_1=2n+1-\lceil 2nt\rceil\right)\right]}{\mathbb{P}\left(\tau_1=2n+1\right)}\end{aligned}$$

where we have used the Markov property, and $\mathbb{P}_{S_0}$ is the probability when the SRW is started at S_0.

Lemma 36. $\mathbb{P}_l(\tau_1=m)=\frac{l+1}{m}\mathbb{P}_l(S_m=-1)$.

Proof. $\mathbb{P}_l(\tau_1=m)=\mathbb{P}_l\left\{S_i\ge 0 \text{ for } 0\le i\le m-1,\ S_{m-1}=0,\ S_m=-1\right\}=\frac{1}{2}\mathbb{P}_l\left\{S_i\ge 0 \text{ for } 0\le i\le m,\ S_{m-1}=0\right.$
so

$$\mathbb{P}_l(\tau_1=m)=\frac{1}{2}\left(\mathbb{P}_l\left\{S_{m-1}=0\right\}-\mathbb{P}_l\left\{S_{m-1}=0,\ \exists i: S_i<0\right\}\right)$$

□

To estimate $\mathbb{P}_l(S_m=-1)$ we use the *local limit theorem*

Fact 37. $\sup_{l\in\mathbb{Z}}\left|\sqrt{m}\mathbb{P}_0\left(S_m=l\vee S_m=l+1\right)-2p_1\left(\frac{l}{\sqrt{m}}\right)\right|\xrightarrow[m\to\infty]{}0$, *where* $p_1(x)$ *is the probability density of the standard Gaussian.*

It follows that $\mathbb{P}_0\left(\tau_1=2n+1\right)=\frac{\sqrt{2n+1}}{(2n+1)^{3/2}}\mathbb{P}_0\left(S_{2n+1}=-1\right)\sim\frac{1}{(2n)^{3/2}}\times 2p_1(0)\sim\frac{1}{2\sqrt{\pi}n^{3/2}}$, and therefore that

$$\mathbb{P}_0\left(\tau_1>\lceil 2nt\rceil\right)\sim\frac{1}{\sqrt{\pi t}n^{1/2}}\,.$$

With the calculation above this gives:

$$\begin{aligned}\mathbb{E}\left[F\left(s\mapsto\frac{C_n(2ns)}{\sqrt{2n}}\right)_{0\le s\le t}\right] &\sim \frac{n}{\sqrt{t}}\mathbb{E}\left[F\left(s\mapsto\frac{S_{2ns}}{\sqrt{2n}}\right)_{0\le s\le t}\frac{S_{\lceil 2nt\rceil}+1}{2n-\lceil 2nt\rceil+1}\mathbb{P}_{S_{\lceil 2nt\rceil}}\left(\tau_1=2n+1-\lceil 2nt\rceil\right)\Big|\tau_1>\right.\\ &\sim \frac{n\sqrt{2n}}{\sqrt{t}\left(2n(1-t)\right)^{3/2}}\mathbb{E}\left[F\left(s\mapsto\frac{S_{2ns}}{\sqrt{2n}}\right)_{0\le s\le t}\frac{S_{\lceil 2nt\rceil}+1}{\sqrt{2n}}\sqrt{2n-2nt}\mathbb{P}_{S_{\lceil 2nt\rceil}}\left(\tau_1=\right.\right.\\ &\sim \frac{n\sqrt{2n}}{\sqrt{t}(2n)^{3/2}}\mathbb{E}\left[F\left(s\mapsto\frac{S_{2ns}}{\sqrt{2n}}\right)_{0\le s\le t}q_{1-t}\left(\frac{S_{2nt}}{\sqrt{2n}}\right)\Big|\tau_1>\sqrt{2nt}\right],\end{aligned}$$

where in the last claim we again used the local limit theorem.

Now recall that, with $e^{(R_t)}$ being the first excursion of BM with duration at least t,

$$\left(s\mapsto\frac{S_{2ns}}{\sqrt{2n}}\right)_{0\le s\le\frac{\tau_1}{2n}}\quad\text{conditionally given }\tau_1>2nt\xrightarrow[n\to\infty]{(d)}e^{(R_t)}\,.$$

It follows that

$$\mathbb{E}\left[F\left(s\mapsto\frac{C_n(2ns)}{\sqrt{2n}}\right)_{0\le s\le t}\right]\xrightarrow[n\to\infty]{}\frac{1}{\sqrt{t}}n\left(F(e(s))_{0\le s\le t}q_{1-t}(e(t))\Big|\zeta(e)>t\right)$$

and since $n(\zeta(e)>t) \sim \frac{1}{\sqrt{2\pi t}}$, we see that

$$\mathbb{E}\left[F\left(s \mapsto \frac{C_n(2ns)}{\sqrt{2n}}\right)_{0\le s\le t}\right] \xrightarrow[n\to\infty]{} \sqrt{2\pi}n\left(F\left(e(s), 0\le s\le t\right) q_{1-t}(e(t))\mathbb{1}_{\{\zeta(e)>t\}}\right)$$
$$= n^{(1)}\left(F\left(e(s), 0\le s\le t\right)\right).$$

Conclusion. Since we fixed $t<1$, we have proved that

$$\left(\frac{C_n(2ns)}{\sqrt{2n}}\right)_{0\le s<1} \xrightarrow[n\to\infty]{(d)} (\mathbf{e}_s)_{0\le s<1}$$

for the topology of uniform convergence over compact sets of $[0,1)$. For the endgame, note that both processes are invariant under reflection: $C_n(2n-\cdot) \overset{(d)}{=} C_n(\cdot)$. It follows that

$$\left(\frac{C_n(2ns)}{\sqrt{2n}}\right)_{0<s\le 1} \xrightarrow[n\to\infty]{(d)} (\mathbf{e}_s)_{0<s\le 1},$$

and together we obtain the Theorem. □

1.5. **Consequences of the scaling limit.** Let $T_n \in \mathbf{T}_n$ be chosen uniformly.

Proposition 38 (Flajolet–Odlyzko 82). *For a rooted tree T let $H(T) = \max\{d_T(root, u) \mid u \in V(T)\}$. Then*

$$\frac{H(T_n)}{\sqrt{2n}} \xrightarrow[n\to\infty]{(d)} \sup_{0\le s\le 1} \mathbf{e}_s .$$

Proof. $C_n(i) = d_{T_n}(\text{root}, u_i)$ where u_i is the ith vertex visited during the contour process. It follows that $H(T_n) = \max_{0\le i\le 2n} C_{T_n}(i) = \max_{0\le s\le 1} C_{T_n}(2ns)$. Since $f \mapsto \sup(f)$ is continuous in the uniform topology on continuous functions, the claim follows. □

Proposition 39. *Let u_n be a uniformly chosen vertex of T_n (conditionally given T_n). Then*

$$\frac{d_{T_n}(root, u_n)}{\sqrt{2n}} \xrightarrow[n\to\infty]{(d)} \mathbf{e}_U ,$$

where U is uniform in $[0,1]$, independent of $\mathbf{e}$.

Remark 40. As seen above, the law of $\mathbf{e}_U$ is the Rayleigh distribution.

Proof. Clear that $\frac{C_{T_n}(2nU)}{\sqrt{2n}} \xrightarrow[n\to\infty]{(d)} \mathbf{e}_U$. What needs to be checked is that $C_{T_n}(2nU)$ is close in distribution to $d_{T_n}(\text{root}, u_n)$, and this can be checked by starting from the contour and forming the tree. □

2. Metric convergence: the continuous random tree

For the Continuous Random Tree and the Brownian Snake, see also [9]

2.1. **Recovering the metric from the contour process.** From the tree T_n we obtain the metric space $\left(V(T_n), \frac{1}{\sqrt{2n}} d_{T_n}\right)$. We would like to interpret Theorem 35 as stating that this random pointed metric space converges in distribution.

For a plane tree $\mathbf{t}$, let u, v be visited in order by the contour process. Suppose that the contour process visits u at time i, v at time j and that $i < j$. Then $\min_{i \le s \le j} C_{\mathbf{t}}(s)$ is the height of the most recent common ancestor of u, v so

$$\begin{aligned} d_{\mathbf{t}}(u,v) &= \left(C_{\mathbf{t}}(i) - \min_{i \le s \le j} C_{\mathbf{t}}(s)\right) + \left(C_{\mathbf{t}}(j) - \min_{i \le s \le j} C_{\mathbf{t}}(s)\right) \\ &= C_{\mathbf{t}}(i) + C_{\mathbf{t}}(j) - 2 \min_{i \le s \le j} C_{\mathbf{t}}(s). \end{aligned}$$

Note that the contour process may visit u twice, but between the visits it remains in the subtree of descendants of u, so $C(s) \ge C(i)$ at that time, and hence this minimum is not affected by the choice of i, as long as at time i we visit u. Similarly for j, and the formula on the right is symmetric in i, j so we can also drop the order assumption.

2.2. **Getting an $\mathbb{R}$-tree from a continuous function.**

Remark 41. For a reference on $\mathbb{R}$-trees see [8]

Given $f \in \mathcal{E}$ recall that that $f \ge 0$ and the notation $f(0) = f(\zeta(f)) = 0$. Then f induces a pseudometric on $[0, \zeta(f)]$ by

$$d_f(s,t) = f(s) + f(t) - 2 \inf \{f(u) \mid s \wedge t \le u \le s \vee t\} .$$

Notation 42. $\check{f}_{s,t} = \{f(u) \mid s \wedge t \le u \le s \vee t\}$.

Definition 43. Let $T_f = ([0, \zeta(f)] / \sim, d_f / \sim)$ be the quotient metric space ($s \sim t$ if $d_f(s,t) = 0$).

Lemma 44. *Note that $p_f \colon [0, \zeta(f)] \to T_f$ be the obvious projection. Then this is continuous for the standard topology on $[0,1]$ and the metric topology of T_f.*

Proof. $d_f(p_f(s), p_f(t)) = d_f(s,t) = f(s) + f(t) - 2\check{f}_{s,t}$. It follows that $\lim_{t \to s} d_f(p_f(s), p_f(t)) = 2f(s) - 2f(s) = 0$. □

Notation 45. Write $\rho_f = p_f(0)$ and call it the *root* of T_f.

We now identify some geodesics in T_f. For this note that $d_f(0,s) = f(s)$ so the distance from the root to $p_f(s)$ is $f(s)$. For $s \in [0, \zeta(s)]$ and $r \in [0, f(s)]$ set

$$\begin{aligned} \Gamma_s^-(r) &= \sup \{t \ge s \mid f(t) = r\} \\ \Gamma_s^+(r) &= \inf \{t \le s \mid f(t) = r\} . \end{aligned}$$

Since $f(t) \ge r$ for all $t \in [\Gamma_s^-(r), \Gamma_s^+(r)]$, $p_f(\Gamma_s^-(r)) = p_f(\Gamma_s^+(r))$ and we call this common value $\gamma_s(r)$.

Lemma 46. *$\gamma_s \colon [0, f(s)] \to T_f$ is a geodesic connecting $0, p_f(s)$.*

Proof. Let $0 \le r \le r' \le f(s)$. Then for every $u \in [\Gamma^+(r'), \Gamma_s^+(r)]$, $f(u) \ge r = f(\Gamma_s^+(r))$. It follows that $d_f(\Gamma^+(r'), \Gamma_s^+(r)) = r + r' - 2r = |r' - r|$. □

Lemma 47. *Let $0 \le s \le t \le \zeta(f)$, and let $u \in [s,t]$ be such that $f(u) = \check{f}_{s,t}$. Then a geodesic from $a = p_f(s)$ to $b = p_f(t)$ is obtained by connecting the pieces of geodesics to ρ that lie above $c = p_f(u)$.*

Proposition 48. *T_f is an $\mathbb{R}$-tree.*

Proof. See [9, Thm. 2.2] □

2.3. **Gromov–Hausdorff convergence.**

Remark 49. For a reference on metric geometry [6, 5]

Definition 50. Let (Z,δ) be a metric space. The *Hausdorff distance* between $A,B\subset Z$ is

$$\delta_{\mathrm{H}}\left(A,B\right)=\sup_{a\in A}\delta(a,B)+\sup_{b\in B}\delta(A,b)\,.$$

This defines a pseudometric.

Definition 51. Let (X,d), (X',d') be compact metric spaces. Set

$$d_{\mathrm{GH}}\left(X,X'\right)=\inf\left\{\delta_Z\left(f(X),f'(X')\right)\mid(Z,\delta)\text{ metric space, }f\colon X\to Z,\ f'\colon X'\to Z\text{ isometric embeddings}\right\}.$$

Example 52. If X is a Euclidean isosceles triangle, X' a point, then $d_{\mathrm{GH}}\left(X,X'\right)=\frac{1}{2}$, where $Z=X\amalg X'$ and $\delta(x,x')=\frac{1}{2}$ for all $x\in X$, where $X'=\{x'\}$.

Proposition 53. *There is a set $\mathbb{M}$ of non-isometric compact metric spaces containing a representative of every isometry class of such metric spaces (this is since every compact metric space has a countable dense subset). Then $(\mathbb{M},d_{GH})$ is a compact metric space.*

Definition 54. A subset $R\subset X\times X'$ is a *correspondence* if for all $x\in X$, $x'\in X'$ we have $\{x\}\times X'\cap R\neq\emptyset$ and $X\times\{x'\}\cap R\neq\emptyset$. The *distortion* of R is the number

$$\mathrm{dist}\left(R\right)=\sup_{\substack{(x,x')\in R\\(y,y')\in R}}\left|d(x,y)-d'(x',y')\right|\,.$$

Proposition 55. *Let (X,d), (X',d')*

Remark 56. For pointed spaces, set

$$d_{\mathrm{GH}}\left(X,X'\right)=\inf\left\{\delta_Z\left(f(X),f'(X')\right)+\delta_Z(f(x),f(x')\mid f\colon(X,d)\to(Z,\delta),\ f'\colon(X',d')\to(Z,\delta)\text{ isom embedding}\right.$$

Then the claims above still hold, with $\mathbb{M}$ the space of pointed compact metric spaces. For the formula with correspondences, insist also that $(x,x')\in R$ where x,x' are the base points.

Proposition 57. *The map $\mathcal{E}\to\mathbb{M}$ given by $f\mapsto(T_f,d_f,\rho_f)$ is 2-Lipschitz.*

Proof. Given $f,g\in\mathcal{E}$ let $R=\{(p_f(t\wedge\zeta(f)),p_g(t\wedge\zeta(g)))\mid t\geq0\}\subset T_f\times T_f$. Then

$$\begin{aligned}\mathrm{dist}(R)&=\sup_{s,t\geq0}\left|d_f\left(s\wedge\zeta(f),t\wedge\zeta(f)\right)-d_g\left(s\wedge\zeta(g),t\wedge\zeta(g)\right)\right|\\&=\sup_{s,t\geq0}\left|\left[f(s\wedge\zeta(f))-g(s\wedge\zeta(g))\right]+\left[f(t\wedge\zeta(f))-g(t\wedge\zeta(g))\right]-2\left(\check{f}_{s,t}-\check{g}_{s,t}\right)\right|\\&\leq4d_{\mathcal{E}}(f,g)\,.\end{aligned}$$

□

2.4. **The Theorem.**

Theorem 58 (Duquesne–Le Gall)**.** *Let T_n be a uniform element of $\mathbf{T}_n$. Then*

$$\left(V(T_n), \frac{d_{T_n}}{\sqrt{2n}}, root\right) \xrightarrow[n\to\infty]{(d)} (T_{\mathbf{e}}, d, \rho)$$

Definition 59. The random metric space $(T_{\mathbf{e}}, d, \rho)$ is called the (Brownian) Continuous Random Tree.

Remark 60. The CRT was introduced by Aldous [1] with a different construction. The construction above is due to Aldous [2] and Le Gall.

Proof of Theorem 58. From the fact that $\frac{C_n(2n\cdot)}{\sqrt{2n}} \xrightarrow[n\to\infty]{(d)} \mathbf{e}$ and the continuity of $f \mapsto T_f$ we get

$$T_{\frac{C_n(2n\cdot)}{\sqrt{2n}}} \xrightarrow[n\to\infty]{(d)} (T_{\mathbf{e}}, d, \rho) \ .$$

Moreover, if the contour process visits $u_i, u_j \in V(T_n)$ at times i, j respectively then $d_{T_n}(u_i, u_j) = d_{C_n}(i, j)$. It follows that $(V(T_n), d_{T_n}, \text{root})$ is isometrically embedded in $(T_{C_n}, d_{C_n}, \text{root})$. Moreover, T_{C_n} is within distance 1 of the image (for all s, $d_{C_n}(s, \lfloor s \rfloor) \leq 1$). It follows that

$$d_{\text{GH}}\left(\left(V(T_n), \frac{d_{T_n}}{\sqrt{2n}}, \text{root}\right), \left(T_{\frac{C_n(2n\cdot)}{\sqrt{2n}}}, \text{distance}, \text{root}\right)\right) \leq \frac{1}{\sqrt{2n}}$$

and we are done. □

Remark 61. Theorem 58 is strictly weaker than Theorem 35, since we have only retained the metric structure of the tree but have lost *the embedding in the plane*, that is the cyclic order of the vertices.

2.5. **Applications.**

Proposition 62. *The CRT has almost surely infinite total length.*

Proof. $\mathbf{e}$ has infinite total variation almost surely. In fact, $\mathbf{e}$ is $\left(\frac{1}{2} - \epsilon\right)$-Hölder continuous for all $\epsilon > 0$ but not $\frac{1}{2}$-Hölder continuous, almost surely (this is inherited from the analogous property of Brownian motion). □

Proposition 63 (Re-rooting principle)**.** *Fix $s \in [0, 1]$. Then $(T_{\mathbf{e}}, d_{\mathbf{e}}, p_{\mathbf{e}}(s)) \overset{(d)}{=} (T_{\mathbf{e}}, d_{\mathbf{e}}, p_{\mathbf{e}}(0))$.*

Proof. The same is true for T_n: there is an obvious automorphism of T_n obtained by re-rooting at the ith edge seen during the contour process. It follows that $\left(V(T_n), d_{T_n}, e_i^-\right) \overset{(\text{d})}{=} \left(V(T_n), d_{T_n}, e_0^-\right)$. To complete the proof choose i_n so that $\frac{i_n}{\sqrt{2n}} \xrightarrow[n\to\infty]{} s \in [0, 1]$ apply the main Theorem. □

Corollary 64. *If U is uniform in $[0, 1]$, independent of $\mathbf{e}$ then $(T_{\mathbf{e}}, d_{\mathbf{e}}, p_{\mathbf{e}}(U)) \overset{(d)}{=} (T_{\mathbf{e}}, d_{\mathbf{e}}, p_{\mathbf{e}}(0))$.*

Thus, if we let $m_{\mathbf{e}}$ be the pushforward to $T_{\mathbf{e}}$ via $p_{\mathbf{e}}$ of Lebesgue measure on $[0, 1]$ then (conditionally given $\mathbf{e}$).

Theorem 65 (Duquesne–Le Gall)**.** *The measure $m_{\mathbf{e}}$ can be recovered from $T_{\mathbf{e}}$: it is the Hausdorff measure on this space with respect to a particular gauge function ($x^2 \log\log \frac{1}{x}$) with a deterministic scaling.*

Corollary 66. *Let U, V are independent and uniform in $[0,1]$ then*$d_{\mathbf{e}}(U,V) \overset{(d)}{=} d_{\mathbf{e}}(U,0) = \mathbf{e}_U$. *In other words, if $x_1, x_2 \in T_{\mathbf{e}}$ are chosen independently according to the measure $m_{\mathbf{e}}$ then $d_{\mathbf{e}}(x_1,x_2) \overset{(d)}{=} d_{\mathbf{e}}(x_1,\rho)$.*

Proposition 67. *Almost surely* $\dim_{\mathrm{H}}(T_{\mathbf{e}}, d_{\mathbf{e}}) \geq 2$.

Proof. However, we know that $\mathbb{P}(2\mathbf{e}_U \in \mathrm{d}x) = xe^{-x^2/2}\,\mathrm{d}x$ so

$$\mathbb{E}m_{\mathbf{e}}\left(B_{d_{2\mathbf{e}}}(\rho,r)\right) = \int_0^r xe^{-\frac{x^2}{2}}\,\mathrm{d}x \sim_{r\to 0} \frac{r^2}{2}\,.$$

By re-rooting we see that the same holds for any ball, and therefore

$$\begin{aligned}\mathbb{E}\left[\int_{T_f} \mathrm{d}m_{\mathbf{e}}(x)\mathbb{1}_{\left\{m_{\mathbf{e}}\left(B_{d_{2\mathbf{e}}}(x,r)\right)>r^{2-\epsilon}\right\}}\right] &\overset{\text{Markov}}{\leq} r^{\epsilon-2}\mathbb{E}\left[\int_{T_f}\mathrm{d}m_{\mathbf{e}}(x)\cdot m_{\mathbf{e}}\left(B_{d_{2\mathbf{e}}}(x,r)\right)\right] \\ &= r^{\epsilon-2}\mathbb{E}\left[m_{\mathbf{e}}B_{d_{2\mathbf{e}}}(\rho,r)\right] \sim_{r\to 0} \frac{r^{\epsilon}}{2}\,.\end{aligned}$$

Now choose $r = 2^{-k}$, $k \geq k_0$. The set of probabilities is then summable, so by Borel–Cantelli, we see that $Prob$-a.s. the set of x for which $m_{\mathbf{e}}\left(B_{d_{2\mathbf{e}}}(x,r)\right) > \left(2^{-k}\right)^{2-\epsilon}$ for infinitely many k has $m_{\mathbf{e}}$-measure zero. Consequently, for $m_{\mathbf{e}}$-a.e. $x \in T_{\mathbf{e}}$,

$$\limsup_{r\to 0} \frac{m_{\mathbf{e}}\left(B_{d_{2\mathbf{e}}}(x,r)\right)}{r^{2-\epsilon}} \leq C < \infty$$

where C is a non-random constant. □

Lemma 68. *Almost surely, the projection $p_{\mathbf{e}}\colon [0,1] \to T_{\mathbf{e}}$ is Hölder continuous of order $\frac{1}{2} - \epsilon$.*

Proof. $d_{\mathbf{e}}(s,t) = (\mathbf{e}(s) - \check{\mathbf{e}}_{s,t}) + (\mathbf{e}(t) - \check{\mathbf{e}}_{s,t}) \leq |\mathbf{e}_s - \mathbf{e}_u| + |\mathbf{e}_t - \mathbf{e}_u|$ for some u between s, t. It follows that

$$d_{\mathbf{e}}(s,t) \leq 2\left\|\mathbf{e}\right\|_{C^{0,\alpha}} |t-s|^{\alpha}\,.$$

□

Corollary 69. *Almost surely,* $\dim_{\mathrm{H}}(T_{\mathbf{e}}, d_{\mathbf{e}}) \leq 2$.

Proof. If $f\colon X \to Y$ is surjective and Hölder continuous of order α then $\dim_{\mathrm{H}} Y \leq \frac{1}{\alpha}\dim_{\mathrm{H}} X$ by pushing forward arbitrary covers of X to Y via f. □

In summary, we obtain:

Theorem 70. *Almost surely,* $\dim_{\mathrm{H}}(T_{\mathbf{e}}, d_{\mathbf{e}}) = 2$.

Remark 71. Recall that the same holds for the trace of Brownian motion in $\mathbb{R}^d$, $d \geq 2$.

3. Decorated trees

3.1. Combinatorics.

Definition 72. Let $\mathbf{t}$ be a a tree. An *admissible labeling* on $\mathbf{t}$ is a 1-Lipschitz function $\ell\colon V(\mathbf{t}) \to \mathbb{Z}$ such that $\ell(\text{root}) = 0$. Write $\mathbb{T}_n$ for the set of labeled trees on n edges.

Lemma 73. $\#\mathbb{T}_n = 3^n \#\mathbf{T}_n = \frac{3^n}{n+1}\binom{2n}{n}$

Proof. DFS the tree. Whenever we visit a new vertex we have 3 choices for the label (given the label of its parent, which is already known). □

Now let $(\mathbf{t}_n, \ell_n) \in \mathbb{T}_n$. $\mathbf{t}_n$ may be encoded using its contour process $C_n = C_{\mathbf{t}_n}$. If u_i is the vertex visited by the contour process at time i then let $L_n(i) = \ell_n(u_i)$. We call L_n the "label process" of the labeled tree. It is important to realize that when the tree is chosen at random L_n is not a nice process. For example, every vertex is visited twice by the contour process and at both times L_n must take the same value.

We will now consider random labeled trees and try to work out their scaling limit. Since every tree in $\mathbf{T}_n$ has the same number (3^n) of labelings, a uniform element $(\mathbf{t}_n, \ell_n) \in \mathbb{T}_n$ is given by choosing $\mathbf{t}_n \in \mathbf{T}_n$ uniformly and choosing ℓ_n uniformly, conditioned in $\mathbf{t}_n$. Note that, conditioned on $\mathbf{t}_n$, the labels on a given simple path are then given by a lazy standard random walk on $\mathbb{Z}$. In particular, the labels of the tree are a tree-indexed random walk, and their limit should be an $\mathbb{R}$-tree indexed Brownian motion.

To identify the scaling exponent note that a one-dimensional SRW reaches distance about $\sqrt{t}$ at time t. Since the branches of a random tree on n vertices have length about $\sqrt{n}$, we will need to scale the labels by $n^{1/4}$.

3.2. **Brownian labels on $\mathbb{R}$-trees.** Let $f \in \mathcal{E}$ and let (T_f, d_f, ρ_f) be the associated $\mathbb{R}$-tree. We'd like to define a process $\left\{Z_a^f\right\}_{a \in T_f}$ so that on each path $[\rho_f, a]$ we see Brownian motion, but for $a, b \in T_f$ the restriction $[\rho_f, a \wedge b]$ are equal while the restrictions to $[a \wedge b, a], [a \wedge b, b]$ are independent.

Proposition 74. *Let Z_s^f be the centered Gaussian process on $[0,t]$ with covariance function* $\operatorname{Cov}\left(Z_s^f, Z_t^f\right) = \check{f}_{s,t}$. *Then, almost surely, Z^f is defined on T_f and has this property.*

Proof. We need to check first that $\check{f}_{s,t}$ is a positive definite kernel. Indeed, for any continuous $g\colon [0, \zeta(f)] \to \mathbb{C}$ we have

$$\begin{aligned}
\iint_{(s,t)\in[0,\zeta(f)]^2} g(s)\overline{g(t)}\check{f}_{s,t}\,\mathrm{d}s\,\mathrm{d}t &= \iint_{(s,t)\in[0,\zeta(f)]^2} g(s)\overline{g(t)}\int_0^\infty \mathrm{d}x \mathbb{1}_{\{x<\check{f}_{s,t}\}}\,\mathrm{d}s\,\mathrm{d}t \\
&\overset{\text{Fubini}}{=} \int_0^\infty \mathrm{d}x \iint_{(s,t)\in[0,\zeta(f)]^2} g(s)\overline{g(t)}\mathbb{1}_{\{x<\check{f}_{s,t}\}}\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^\infty \mathrm{d}x \sum_{I\in\mathcal{I}(x)} \iint_{(s,t)\in I^2} g(s)\overline{g(t)}
\end{aligned}$$

where $\mathcal{I}(x)$ are the connected components of the set $\{u \mid f(u) \geq x\}$. The reason for this is that if s,t belong to different intervals in this set then there is a point between s,t where f dips below x and so $\check{f}_{s,t} < x$, while if they are in the same interval then f does not dip below them so that $\check{f}_{s,t} \geq x$ also. We can rewrite this identity as

$$\iint_{(s,t)\in[0,\zeta(f)]^2} g(s)\overline{g(t)}\check{f}_{s,t}\,\mathrm{d}s\,\mathrm{d}t = \int_0^\infty \mathrm{d}x \sum_{I\in\mathcal{I}(x)} \left|\int_{s\in I} g(s)\,\mathrm{d}s\right|^2 \geq 0\,.$$

To see that Z^f is defined on T_f, let $\mathcal{I}^\circ(x)$ be the connected components of the open set $\{u \mid f(u) > x\}$. This is a disjoint union of intervals, and if (a,b) is such an interval then $d_f(a,b)=0$. Let $J=\{(a,b)\in\mathbb{R}^2 \mid \exists q\in\mathbb{Q}:(a,b)\in I(q)\}$. This is a countable set, and for each $(a,b)\in J$ we have $\mathbb{E}\left[\left|Z_a^f-Z_b^f\right|^2\right]=d_f(a,b)=0$; taking a countable union we have a.s. that for every $(a,b)\in J$, $Z_a^f=Z_b^f$. Finally, check that if $d_f(s,t)=0$ then $(s,t)\in\mathbb{R}^2$ is a limit point of J.

Next, let $s\le u\le t\in[0,\zeta(f)]$ with $p_f(u)=p_f(s)\wedge p_f(t)$, that is with $f(u)=\check{f}_{s,t}$.

Finally, we need to show that $Z_s^f-Z_u^f$, $Z_t^f-Z_u^f$ are independent. It suffices to check that they are uncorrelated, and indeed

$$\begin{aligned}\mathbb{E}\left[\left(Z_s^f-Z_u^f\right)\left(Z_t^f-Z_u^f\right)\right] &= \mathbb{E}\left[Z_s^fZ_t^f-Z_s^fZ_u^f-Z_t^fZ_u^f+Z_u^fZ_u^f\right]\\ &= \check{f}_{s,t}-\check{f}_{s,u}-\check{f}_{t,u}+\check{f}_{u,u}\\ &= f(u)-f(u)-f(u)+f(u)\\ &= 0\,.\end{aligned}$$

□

Lemma 75. *Suppose f is Hölder continuous of order $\alpha>0$. Then the process Z^f admits a modification which is almost surely Hölder continuous of any order $\beta\in\left(0,\frac{\alpha}{2}\right)$.*

Proof. By Kolmogorov's criterion it is enough to estimate the moments $\mathbb{E}\left[\left|Z_s^f-Z_t^f\right|^{2p}\right]$. Since $Z_s^f-Z_t^f$ is a centered Gaussian with variance

$$\mathbb{E}\left[\left|Z_s^f-Z_t^f\right|^2\right]=\check{f}_{s,s}+\check{f}_{t,t}-2\check{f}_{s,t}=d_f(s,t)$$

we have

$$\begin{aligned}\mathbb{E}\left[\left|Z_s^f-Z_t^f\right|^{2p}\right] &= \mathbb{E}\left[\mathcal{N}(0,d_f(s,t))^{2p}\right]\\ &= d_f(s,t)^p\,\mathbb{E}\left[\mathcal{N}(0,1)^{2p}\right]\\ &\ll_p \|f\|_{C^{0,\alpha}}\,|s-t|^{\alpha p}\end{aligned}$$

and therefore Z^f can be modified to be $\left(\frac{\alpha p-1}{2p}\right)$-Hölder continuous for every $p>0$. □

From now on we work with the continuous modification.

Corollary 76. *Almost surely for every s,t such that $d_f(s,t)=0$ we have $Z_s^f=Z_t^f$.*

We now specialize to the case where $f=\mathbf{e}$ (recalling that $\mathbf{e}$ is a.s. Hölder continuous of every order $\alpha<\frac{1}{2}$), writing $Z=Z^{\mathbf{e}}$.

Definition 77 (Le Gall)**.** The pair $(\mathbf{e},Z)$ is called the "head of the Brownian snake".

Remark 78. We will need several properties of this process that are best proved using the full "Brownian snake" process, which we will not consider.

3.3. The scaling limit of random labeled trees.

Theorem 79 (Chassing–Schaeffer, Janson–Marckert). *Let $(\mathbf{t}_n, \ell_n)$ be uniform in $\mathbb{T}_n$, and let C_n, L_n be the associated contour and label processes (linearly interpolated in $[0,2n]$). Then*

$$\left(\left(s \mapsto \frac{C_n(2ns)}{\sqrt{2n}}\right), \left(s \mapsto \left(\frac{9}{8n}\right)^{1/4} L_n(2ns)\right)\right) \xrightarrow[n\to\infty]{(d)} (\mathbf{e}, Z)$$

in $\left(C([0,1],\mathbb{R})\right)^2$ with the topology of uniform convergence.

Remark 80. There is an analogous metric convergence in the space of pairs (X, f) where X is a compact metric space and f a bounded function on X,

Proof. We will first show that the finite-dimensional marginals converge, and then establish tightness.

Step 1. We already know that $\frac{C_n(2n\cdot)}{\sqrt{2n}} \xrightarrow[n\to\infty]{(\mathrm{d})} \mathbf{e}$. By the Skorokhod representation Theorem, we can assume that on the underlying probability space this convergence occurs almost surely. Recall that the trees may be recovered from the contour functions. Accordingly, we assume that the probability space further carries a family of random variables $(Y_e^n)_{e\in E(T_n)}$ which are iid on $\{0,1,-1\}$, conditionally given $(T_n)_{n\geq 1}$, and such that the label function L_n satisfies $L_n(u) = \sum_{e\vdash u} Y_e^n$ ($e \vdash u$ means that e is on the path from the root to u). Now, conditionally given $(C_n)_{n\geq 1}$ and $\mathbf{e}$, for each $t\in(0,1)$ we have that

$$\left(\frac{9}{8n}\right)^{1/4} L_n(2nt) \xrightarrow[n\to\infty]{(\mathrm{d})} \mathcal{N}(0,\mathbf{e}_t)\,.$$

Indeed, since $|L_n(2nt) - L_n(\lfloor 2nt\rfloor)| \leq 1$ we can use $L_n(\lfloor 2nt\rfloor)$ instead. This is $\sum_{e\vdash u_{\lfloor 2nt\rfloor}} Y_e^n$, a sum of $C_n(\lfloor 2nt\rfloor)$ iid centered random variables. Since $\mathrm{Var}(Y_e^n) = \frac{2}{3}$, we have

$$L_n(\lfloor 2nt\rfloor) \overset{(\mathrm{d})}{=} \sum_{i=1}^{C_n(\lfloor 2nt\rfloor)} \tilde{Y}_i = \frac{\sum_{i=1}^{C_n(\lfloor 2nt\rfloor)} \tilde{Y}_i}{\sqrt{\frac{2}{3}C_n(\lfloor 2nt\rfloor)}} \times \sqrt{\frac{2}{3}C_n(\lfloor 2nt\rfloor)}$$

where $\tilde{Y}_i$ are iid uniform in $\{0,\pm 1\}$. Now using the central limit theorem and the convergence of $\frac{C_n}{\sqrt{2n}}$ we have:

$$\begin{aligned}
\frac{L_n(\lfloor 2nt\rfloor)}{(8n/9)^{1/4}} &\overset{(\mathrm{d})}{=} \left(\frac{\sum_{i=1}^{C_n(\lfloor 2nt\rfloor)} \tilde{Y}_i}{\sqrt{\frac{2}{3}C_n(\lfloor 2nt\rfloor)}}\right) \times \left(\sqrt{\frac{C_n(\lfloor 2nt\rfloor)}{\sqrt{2n}}}\right) \\
&\xrightarrow[n\to\infty]{(\mathrm{d})} \mathcal{N}(0,1) \times \sqrt{\mathbf{e}_t} \\
&\overset{(\mathrm{d})}{=} \mathcal{N}(0,\mathbf{e}_t)\,.
\end{aligned}$$

We next consider the convergence of two-point marginals. For this let $0 < t_1 < t_2 < 1$. We want to show that, given $(C_n)_{n\geq 1}$,

$$\left(\frac{9}{8n}\right)^{1/4} \left(L_n(\lfloor 2nt_1\rfloor), L_n(\lfloor 2nt_2\rfloor)\right) \xrightarrow[n\to\infty]{(\mathrm{d})} \left(Z_{t_1}, Z_{t_2}\right)\,.$$

Note that, given $\mathbf{e}$, (Z_{t_1}, Z_{t_2}) is a centered Gaussian vector with covariance matrix $\begin{pmatrix} \mathbf{e}_{t_1} & \check{\mathbf{e}}_{t_1,t_2} \\ \check{\mathbf{e}}_{t_1,t_2} & \mathbf{e}_{t_2} \end{pmatrix}$. Write $\check{u}_n$ for the last common ancestor of $u_{\lfloor 2nt_1 \rfloor}, u_{\lfloor 2nt_2 \rfloor}$ in T_n. Then

$$L_n\left(\lfloor 2nt_i \rfloor\right) = A^n + B_i^n$$

where $A^n = \sum_{e \vdash \check{u}_n} Y_e^n$ and

$$B_i^n = \sum_{\check{u}_n \vdash e \vdash u_{\lfloor 2nt_i \rfloor}} Y_e^n .$$

Also, the three variables A^n, B_1^n, B_2^n are independent, and each is a sum of some number of iid $\tilde{Y}$. Since the distance of $\check{u}_n$ to the root is $\check{C}_n\left(\lfloor 2nt_1 \rfloor, \lfloor 2nt_2 \rfloor\right)$ we can apply the same argument as for the one-point function to conclude that

$$\left(\frac{9}{8n}\right)^{1/4} \left(A^n, B_1^n, B_2^n\right) \xrightarrow[n\to\infty]{(\mathrm{d})} \left(A, B_1, B_2\right)$$

where on the RHS we have three independent Gaussians with respective variances $\check{\mathbf{e}}_{t_1,t_2}$, $\mathbf{e}_{t_1} - \check{\mathbf{e}}_{t_1,t_2}$, $\mathbf{e}_{t_2} - \check{\mathbf{e}}_{t_1,t_2}$. It follows that

$$\left(\frac{9}{8n}\right)^{1/4} \left(A^n + B_1^n, A^n + B_2^n\right) \xrightarrow[n\to\infty]{(\mathrm{d})} \left(A + B_1, A + B_2\right) \overset{(\mathrm{d})}{=} \left(Z_{t_1}, Z_{t_2}\right) .$$

Now suppose we have $0 < t_1 < t_2 < \cdots < t_k < 1$. Drawing the subtree of T_n spanned by those points (whose combinatorial type eventually does not depend on n by convergence of the C_n) one can write random variables A_n (corresponding to the most recent common ancestor of all times) and B_p^n (corresponding to the paths between various ancestors in the tree) so that these are all independent sums of iid $\tilde{Y}$ of lengths that depend on the contour process so that $L_n(\lfloor 2nt_i \rfloor)$ can be written as the sum of A^n and those B_p^n so that the paths together add up to a path from the root to the appropriate vertex. Now the by the CLT and the convergence of $(C_n)_{n\geq 1}$ the scaled A^n, B_p^n will converge to independent Gaussians A, B_p, with variances given by values to $\mathbf{e}$, and it will follow that the L_n converge to vector of the Z_{t_i}.

Tightness. By Kolmogorov's criterion it is enough to show that, for all $0 \leq s < t \leq 1$,

$$\mathbb{E}\left[\left|\frac{L_n(\lfloor 2ns \rfloor) - L_n(\lfloor 2nt \rfloor)}{n^{1/4}}\right|^{2p} \,\middle|\, \left(C_n\right)_{n\geq 1}\right] \leq C_p(\mathbf{e})\left|t - s\right|^p$$

where $C_p(\mathbf{e})$ is almost surely finite. Indeed, for $0 \leq i < j \leq 2n$ we have

$$\mathbb{E}\left[\left|L_n(i) - L_n(j)\right|^{2p} \,\middle|\, \left(C_n\right)_{n\geq 1}\right] = \mathbb{E}\left[\left|\sum_{i=1}^{d_{T_n}(u_i,u_j)} \tilde{Y}_i\right|^{2p}\right] \leq K_p d_{T_n}(u_i, u_j)^{2p}$$

by a quantitative CLT. Writing the distance in terms of C_n we find that, unconditionally,

$$\mathbb{E}\left[\left|L_n(i) - L_n(j)\right|^{2p}\right] \leq K_p \mathbb{E}\left[\left|C_n(i) + C_n(j) - 2\check{C}(i,j)\right|^{2p}\right] .$$

By the re-rooting invariance of the random tree, the expectation in the RHS is equal to $\mathbb{E}\left[\left|C(j-i)\right|^{2p}\right] = \sum_{l=0}^{\infty} \mathbb{P}\left(C_n(j-i) = l\right) l^{2p}$, and the latter can be estimated by counting paths. □

Part 2. Random maps

4. The Cori–Vanquelin–Schaeffer bijection

Our first goal is obtaining the following enumeration.

Theorem 81 (Tutte)**.** *There are* $\frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}$ *rooted planar maps with* n *edges.*

Recall that a planar map is a *quadrangulation* if all its faces have degree 4.

4.1. **The "trivial" bijection.** Start with be a planar map $\mathbf{m}$. Add a (dual) vertex in each face, and connect each of those vertices to every "corner" in boundary of the corresponding face (there are in bijection with the oriented edges along the boundary by associating to each oriented edge its tail vertex. Note that this is not the same as vertices on the boundary, since a vertex can appear several times in the boundary cycle of an edge. Now every undirected edge is the union of two oriented edges, each connected to a single dual vertex. Removing the two oriented edges we are left with a quadrangle, whose vertices are the two endpoints of the edge and the two dual vertices at the two sides (these don't have to be distinct).

This process converts a planar map with n edges to a quadrangulation with n faces, together with a 2-coloring of the vertices (we know which vertices of the quadrangulation come from vertices of the map, which from dual vertices). We move the root by fixing its source vertex and moving the endpoint to the next vertex of the quadrangle formed from the original root edge.

For the converse we need the following:

Lemma 82. *Planar quadrangulations are bipartite graphs.*

Proof. A simple cycle in a plane map is a Jordan curve and divides the sphere into two parts; apply Euler's formula to one of those parts. □

Now given a rooted planar map $\mathbf{q}$, the root edge gives a choice of one side of the bipartition (the side containing its tail). We then let $\mathbf{m}$ be the planar map with vertices being the vertices on that side, and edges being those "diagonals" of the quadrangles which connect vertices in the chosen side.

Corollary 83. *Let* $\mathbf{Q}_n = \{$*rooted plane quadrangulations with* n *faces*$\}$, $\mathbf{Q}_n^\bullet = \{(\mathbf{q}, v_*) \mid \mathbf{q} \in \mathbf{Q}_n,\ v_* \in V(\mathbf{q})\}$. *Then Tutte's Theorem 81 is equivalent to the assertion* $\#\mathbf{Q}_n^\bullet = 2\frac{3^n}{n+1}\binom{2n}{n}$.

Proof. First, since *rooted* plane maps have no automorphisms and every quadrangulation with n faces has $n+2$ vertices (apply Euler's formula, using $4\#F(\mathbf{q}) = 2\#E(\mathbf{q})$), $\#\mathbf{Q}_n^\bullet = (n+2)\#\mathbf{Q}_n$. Second, the bijection just constructed shows that $\#\mathbf{Q}_n$ is the number of plane maps. □

Lemma 84. *Let* G *be a connected bipartite graph, and let* $u, v, w \in V(G)$ *with* u, v *adjacent. Then* $|d_G(w,u) - d_G(w,u)| = 1$.

4.2. **The C–V–S bijection.** Given $(\mathbf{q}, v_*) \in \mathbf{Q}_n^\bullet$ let $\bar{\ell}\colon V(\mathbf{q}) \to \mathbb{Z}$ be the distance to v_*. By the last Lemma, if u, v are adjacent $\left|\bar{\ell}(u) - \bar{\ell}(v)\right| = 1$. It follows that the labels around a face must have one of the patterns $(\ell, \ell+1, \ell+2, \ell+1)$ (in cyclic order around the face) or $(\ell, \ell+1, \ell, \ell+1)$ (in cyclic order). Construct a map $\mathbf{t}$ consisting of the vertices of $\mathbf{q}$ and new edges as follows: for each face of the first type take an edge between the second and third vertices inside the face, in the second case the two opposite vertices with label $\ell+1$.

Claim 85. $\mathbf{t}$ is a tree (as yet, unrooted).

Proof. We show that $\#V(\mathbf{t}) = n+1$, that $\#E(\mathbf{t}) = n$, and that $\mathbf{t}$ has no cycles.

First, let $v \neq v_*$ be any vertex. Then $d(v,v_*) > 0$ and there is a neighbour v' of v with $\ell \stackrel{\text{def}}{=} d(v',v_*) = d(v,v_*) - 1$. Let e be the directed edge from v to v', and let f be the face *to the right* of e. The labels along e' are $\ell+1$ (at tail) and ℓ (at head). There are several possibilities for the labels of the other two vertices in that face, but one checks that in any case the $\mathbf{t}$-edge associated to this face is incident to v, the tail of e. It follows that every $v \in V(\mathbf{q}) \setminus \{v_*\}$ is incident to at least one edge in tt. Since v_* itself is not incident to an edge we have $\#V(\mathbf{t}) = n+1$. By construction we have $\#E(\mathbf{t}) = \#F(\mathbf{q}) = n$.

Finally, suppose $\mathbf{t}$ had a simple cycle. By the Jordan Theorem this cycle divides the sphere into two connected parts, and v_* belongs to one of them. Let u be a vertex on the cycle with minimal label ℓ. Suppose first that it has a $\mathbf{t}$-neighbour v along the cycle with the same label. Then u,v belong to a quadrangle with labels $\ell-1,\ell,\ell-1,\ell$, with the two vertices with labels $\ell-1$ on different sides of the cycle. This is impossible: consider the vertex with label $\ell-1$ which is in the part not containing v_*– any shortest $\mathbf{q}$-path from v_* to w must cross the cycle and therefore meet a vertex with label at least ℓ, a contradiction. Otherwise, the two $\mathbf{t}$-neighbours of u along the cycle, say v_1, v_2, have labels $\ell+1$. It follows that (v_1,u) and (u,v_2) are edges of $\mathbf{q}$ belonging to faces of $\mathbf{q}$, one inside the cycle and one outside, and each contains a vertex with label $\ell-1$, again a contradiction. □

Remark 86. This was the original proof, which is specific to the sphere via the Jordan Curve Theorem, but there are proofs which generalize to surfaces of any genus.

Thus given $(\mathbf{q},v_*)$ we have a tree $\mathbf{t}$ on $n+1$ vertices, hence n edges. To construct an admissible labeling of this tree, let v_0 be that endpoint of the root e_* with the *larger* label, and set $\ell(v) = \bar{\ell}(v) - \bar{\ell}(v_0)$. Let $\epsilon = \begin{cases} +1 & v_0 = e_*^- \\ -1 & v_0 = e_*^+ \end{cases}$. It is clear from the definition of the edges of $\mathbf{t}$ that ℓ is an admissible labeling of $\mathbf{t}$, if we root $\mathbf{t}$ at the vertex v_0 and the edge begin the edge of $\mathbf{t}$ associated to the face of $\mathbf{q}$ lying *to the right* of that orientation of e_* which beings at v_0.

Theorem 87. *(C–V–S) The map* $\Phi\colon \mathbf{Q}_n^\bullet \to \mathbb{T}_n \times \{\pm 1\}$ *given by* $\Phi(\mathbf{q},v_*) = ((\mathbf{t},\ell),\epsilon)$ *is a bijection.*

Proof. We exhibit an inverse $\Psi\colon \mathbb{T}_n \times \{\pm1\} \to \mathbf{Q}_n^\bullet$ (which is, in fact, the more useful direction of the bijection).

Accordingly, given $((\mathbf{t},\ell),\epsilon)$, let $\{e_j\}_{j=0}^{2n-1}$ be the oriented edges in the contour (DFS) exploration of $\mathbf{t}$,with e_0 being the root edge of $\mathbf{t}$, and extend this by $e_{2n+j} = e_j$. Also, set $\ell(e) = \ell(e^-)$ for each $e \in \vec{E}(\mathbf{t})$.

First, add a vertex v_* not in $\mathrm{supp}(\mathbf{t})$ incident to a single oriented edge e_∞. Next, for each $i \geq 0$ set $s(i) = \inf\{j \geq i \mid \ell(e_j) = \ell(e_i) - 1\} = \inf\{j > i \mid \ell(e_j) < \ell(e_i)\}$ (here $\inf \emptyset = \infty$), and set $s(e_i) = e_{s(i)}$. Now for each $e \in \vec{E}(\mathbf{t})$ draw an arc from e to $s(e)$.

Let $\mathbf{q} = \Psi((\mathbf{t},\ell),\epsilon)$ be the graph with vertices $V(\mathbf{t}) \cup \{v_*\}$ and edges the new arcs, rooted at the edge from the root edge of $\mathbf{t}$ to its successor if $\epsilon = +1$, at the reverse of this edge if $\epsilon = -1$.

Remark 88. The reason for extending $\{e_j\}$ cyclically is that finding successor to a given edge might require going around the tree past the root.

The rest of the proof is subdivided into several claims.

Claim 89. The arcs can be drawn in non-crossing fashion, that is $\mathbf{q}$ is a plane map.

Track the contour process of a plane tree as running on the boundary of a convex polygon with $2n$ sides. Each corner of the tree corresponds to a vertex of the polygon, so that a vertex in the tree of degree d corresponds to d vertices of the polygon, with the identification being such that the convex hulls of the sets of polygon vertices corresponding to different tree vertices are disjoint (one can think of the tree as coming from collapsing the polygon). Now the (single) face of the tree corresponds to the outside face of the polygon.

Next, we label the vertices of the polygon according to the labels from the tree, and consider the edges of $\mathbf{q}$. First, all vertices with minimal label are connected to a new vertex in the outside face. Next, each other vertex is connected to its successor, that being the next vertex of the polygon (going clockwise) with a smaller label. There is no intersection of the edges: suppose a vertex v_i of the polygon is connected in $\mathbf{q}$ to its successor v_j. Now consider v_k with $i < k < j$ in the cyclic order (clockwise around the polygon). Then the label of v_k is at least that of v_i (since it comes before v_j). Since the label of v_j is strictly less than that of v_i, it's also less than the label of v_k, so the successor of v_k occurs before v_i and we can draw the edge leaving v_k without crossing. A similar argument applies if k occurs after j but before i.

Claim 90. $\mathbf{q}$ is a quadrangulation.

Proof. Suppose we have a tree edge e connecting vertices with labels $\ell, \ell+1$ in that order. Let e' be the next edge leaving the vertex of label $\ell+1$. Then $s(e) = s(s(e'))$ and we get a degree-4 face containing the edge. Suppose that e connects vertices with the same label ℓ. Then there is an edge e' leaving the head of e so that $s(e) = s(e')$ has label $\ell - 1$, and similarly e'' so that $s(\bar{e}) = s(e'')$ has label $\ell + 1$, and we obtain a face of $\mathbf{q}$ of degree 4.

This shows that $\mathbf{q}$ has at least n faces of degree 4. But $\mathbf{q}$ has $2n$ edges (one for each corner of the polygon) and exactly $n+2$ vertices (the vertices of the tree plus one) so by Euler's formula exactly n faces. It follows that $\mathbf{q}$ is a quadrangulation. □

Lemma 91 (Key lemma). *Let* $(\mathbf{q}, v_*) = \Psi((\mathbf{t}, \ell), \epsilon)$. *Then for every* $v \in V(\mathbf{q}) \setminus \{v_*\} = V(\mathbf{t})$, $d_{\mathbf{q}}(v, v_*) = \bar{\ell}(v) \overset{def}{=} \ell(v) - \min \ell(V(\mathbf{t})) + 1$ *(could include* v_* *by setting* $\bar{\ell}(v_*)$ *appropriately)*

Proof. Note that $|\ell(u) - \ell(v)| = \left|\bar{\ell}(u) - \bar{\ell}(v)\right| = 1$ if u, v are connected by an arc of $\mathbf{q}$. Since vertices of minimal label are connected to v_*, the triangle inequality gives that $\bar{\ell}(v) \leq d_{\mathbf{q}}(v, v_*)$. For the converse let e be any edge leaving v, and consider the chain $e \to s(e) \to s^2(e) \to s^3(e)$ and so on, which are edges of $\mathbf{q}$. The labels are strictly decreasing along this path, so the length of this chain is at most $\bar{\ell}(v)$ and we have $d_{\mathbf{q}}(v, v_*) \leq \bar{\ell}(v)$. □

Claim 92. Ψ is a right inverse to Φ ($\Phi \circ \Psi = \mathrm{id}$).

Proof. The proof of the claim 90 shows that the edge Φ would associate to every face of $\Psi((\mathbf{t},\ell),\epsilon)$ is precisely the edge of $\mathbf{t}$ corresponding to that face [we have shifted the labels by a constant, but that does not change the property that the vertices of a face have labels like $\ell,\ell+1,\ell,\ell+1$), while the Lemma shows that the labels are the ones Φ uses. One also needs to verify that the rooting convention is recovered. □

Claim 93. Ψ is a left inverse to Φ (proof omitted; see [7, 4]).

□

4.3. **An alternative point of view of the CVS bijection.** Replace every vertex of a planar map $\mathbf{q}$ with a "roundabout": a circle in the plane which meets edges at the vertex at distinct points.

Now cars leave v_* at take unit time to traverse an edge. Flows meeting at vertices follow the *roundabout rule*: if traffic has arrived into the vertex from an edge it travels right (counter-clockwise) and takes exits as long as no traffic has already come in through those edges. Otherwise the traffic is jammed. It is easy to check that eventually every edge will carry traffic in some direction, and that if we only include the arcs of the roundabout that actually carry traffic together with these edges we'll get a spanning tree of $\mathbf{q}$. Also, if we let a bud leave the vertex at each arc of the roundabout where there is no traffic (since traffic is incoming rather than outgoing at the edge to the right) then every face of $\mathbf{q}$ will have exactly two incoming buds which we are connected in a *tree*-edge. One can check what this means in terms of the labels.

- The CVS bijection is thus a discrete analogue to the *cut locus* on a Riemannian manifold (the cut locus of $p\in M$ is the set of points q for which there are at least two minimizing geodesics between p,q).
- But note that while on the sphere the cut locus of a point is exactly the antipodal point, on any quadrangulation of the sphere the discrete cut locus is practically dense, since there are many short cycles.

5. Random quadrangulations

Remark 94. Let $Q_n\in\mathbf{Q}_n$ be uniform. Then if we randomly choose a vertex $v_*\in V(Q_n)$, uniformly conditioned on Q_n, the pair (Q_n,v_*) is a uniform element of $\mathbf{Q}_n^{\bullet}$.

From now on assume that $(Q_n,v_*)=\Phi((T_n,\ell_n),\epsilon)$ where $(T_n,\ell_n)\in\mathbb{T}_n$ is uniform and ϵ is independent and uniform in ±1.

We start by considering basic statistics of $\mathbf{q}$:

Definition 95. The *radius* of $\mathbf{q}$ (as seen from v_*) is

$$\mathcal{R}(\mathbf{q},v_*)=\max\{d_{\mathbf{q}}(u,v_*)\mid u\in V(\mathbf{q})\}\ .$$

The *profile* of $\mathbf{q}$ (as seen from v_*) is the sequence of volumes

$$I_{\mathbf{q},v_*}(k)=\#\{d_{\mathbf{q}}(u,v_*)=k\mid u\in V(\mathbf{q})\}\ .$$

Theorem 96 ((1),(3) in Chassing–Schaeffer 2004, (2) in Le Gall 2005)**.** *Let $\mathbb{Q}_n$ be uniform in $\mathbf{Q}_n$, v_*,v_{**} two independent uniform vertices of Q_n (given Q_n). Let $(\mathbf{e},Z)$ be the head of the Brownian snake. Then:*

(1) $\left(\frac{9}{8n}\right)^{1/4}\mathcal{R}\left(Q_n,v_*\right)\xrightarrow[n\to\infty]{(\mathrm{d})}\sup Z-\inf Z$;

(2) $\left(\frac{9}{8n}\right)^{1/4}d_{\mathbf{q}}\left(v_*,v_{**}\right)\xrightarrow[n\to\infty]{(\mathrm{d})}\sup Z$;

(3) *The random measures*

$$\mu_{(Q_n,v_*)}=\sum_{k\geq 0}\frac{I_{(Q_n,v_*)}(k)}{n+2}\delta_{\left(\frac{9}{8n}\right)^{1/4}k}=\frac{1}{\#V(\mathbf{q})}\sum_{v\in V(\mathbf{q})}\delta_{\left(\frac{9}{8n}\right)^{1/4}d_{\mathbf{q}}(v,v_*)}$$

converge weak- in distribution to the random probability measure on* $\mathbb{R}_{\geq 0}$

$$\mu_{(\mathbf{e},Z)}(g)=\int_0^1 g\left(Z_s-\inf Z\right)\mathrm{d}s$$

Remark 97. The "Integrated super-Brownian excursion" (Aldous 90's) is the related random measure $\mathrm{ISE}(g)=\int_0^1 g(Z_s)\,\mathrm{d}s$.

Proof of Theorem. (1) For the first claim, recall that if $(Q_n,v_*)=\Phi\left((T_n,\ell_n),\epsilon\right)$ then $d_{Q_n}(v,v_*)=\ell_n(v)-\min_{u\in V(T_n)}\ell(u)+1$ so

$$\begin{aligned}\mathcal{R}(Q_n,v_*)&=\max\ell_n-\min\ell_n+1\\&=\max L_n-\min L_n+1\end{aligned}$$

where L_n is the *label process* $L_n(i)=\ell_n(u_i)$. It follows that

$$\left(\frac{9}{8n}\right)^{1/4}\mathcal{R}\left(Q_n,v_*\right)=\max_{s\in[0,1]}\frac{L_n(2ns)}{(8n/9)^{1/4}}-\min_{t\in[0,1]}\frac{L_n(2nt)}{(8n/9)^{1/4}}+\frac{1}{(8n/9)^{1/4}}.$$

Now, since $\frac{L_n(2ns)}{(8n/9)^{1/4}}\xrightarrow[n\to\infty]{(\mathrm{d})}Z$ *in the uniform topology*, in which max, min are continuous, it follows that

$$\left(\frac{9}{8n}\right)^{1/4}\mathcal{R}\left(Q_n,v_*\right)\xrightarrow[n\to\infty]{(\mathrm{d})}\max Z-\min Z+0\,.$$

(2) It suffices to prove the result if v_{**} is uniform in $V(Q_n)\setminus\{v_*,\mathrm{root}\}$ (the root of T_n) since the event $v_{**}\in\{v_*,\mathrm{root}\}$ occurs with probability $\frac{2}{n+2}\to 0$. Now $V(Q_n)\setminus\{v_*,e_*^-\}=V(T_n)\setminus\{\mathrm{root}\}$. We have seen that one way to choose a random element of this set is to let U be uniform in $[0,1]$ and take the vertex $u_{\langle U\rangle_n}$ where

$$\langle U\rangle_n=\begin{cases}\lfloor 2nU\rfloor & \frac{dC_n(2nU)}{dx}=-1\\ \lceil 2nU\rceil & \frac{dC_n(2nU)}{dx}=+1\end{cases},$$

so we may assume that $v_{**}=u_{\langle U\rangle_n}$. In that case,

$$\begin{aligned}d_{Q_n}(v_*,v_{**})&=\ell_n(u_{\langle U\rangle_n})-\min\ell_n+1\\&=L_n(2nU)-\min L_n+1+\left(\ell_n(u_{\langle U\rangle_n})-L_n(2nU)\right).\end{aligned}$$

Since $\left|\ell_n(u_{\langle U\rangle_n})-L_n(2nU)\right|\leq 1$ It follows that

$$\left(\frac{9}{8n}\right)^{1/4}d_{Q_n}(v_*,v_{**})=\left(\left(\frac{9}{8n}\right)^{1/4}L_n(2nU)\right)-\left(\min_t\left(\frac{9}{8n}\right)^{1/4}L_n(2nt)\right)+\left(\frac{9}{8n}\right)^{1/4}X$$

where $|X|\leq 2$, and hence that

$$\left(\frac{9}{8n}\right)^{1/4}d_{Q_n}(v_*,v_{**})\xrightarrow[n\to\infty]{(\mathrm{d})}Z_U-\inf Z\,.$$

It remains to show that $Z_U - \inf Z \overset{(d)}{=} \sup Z$. For this, note that re-rooting T_n at e_i and shifting the labels by $-L_n(i)$ leaves the label measure on the label process invariant. In other words, for i fixed the function $(j \mapsto (L_n(j+i) - L_n(i)))$ has the same distribution as L_n. Taking the limit as $n \to \infty$ we see that $(t \mapsto (Z_{t+s} - Z_s)) \overset{(d)}{=} Z$. Now taking infimum in s and choosing t uniformly gives the claim.

(3) Again, if we integrate $s \in [0,1]$ w.r.t. Lebesgue measure, $u_{\langle s \rangle_n}$ ranges over all the vertices of the tree with equal weights. It follows that

$$\frac{1}{n} \sum_{u \in V(Q_n)} g(u) = \int_0^1 \mathrm{d}s g\left(d_{Q_n}\left(v_*, u_{\langle s \rangle_n}\right)\right) + \frac{1}{n}\left(g(0) + g\left(d_{Q_n}\left(v_*, \mathrm{root}_{T_n}\right)\right)\right) .$$

Rescaling and using $|L_n(2ns) - L_n(\langle s \rangle_n)| \le 1$ we see that

$$\mu_{(Q_n, v_*)}(g) = \int_0^1 \mathrm{d}s g\left(\frac{L_n(2ns) - \min L_n}{(8n/9)^{1/4}}\right) + o_g(1)$$

and hence that

$$\mu_{(Q_n, v_*)}(g) \xrightarrow[n\to\infty]{(d)} \int_0^1 \mathrm{d}s g\left(Z_s - \min Z\right) .$$

□

Exercise 98. (Specific vertices)

(1) Show that $\frac{\mathcal{R}\left(Q_n, e_*^-\right)}{(8n/9)^{1/4}} \xrightarrow[n\to\infty]{(d)} \sup Z - \inf Z$.
Hint: Show that if e_{**} is uniform in $\vec{E}(Q_n)$ then Q_n re-rooted at e_{**} has the same law as Q_n.

(2) Show that $\frac{d_{Q_n}\left(v_*, e_*^-\right)}{(8n/9)^{1/4}} \xrightarrow[n\to\infty]{(d)} \sup Z$.
Hint: Couple e_*^- with a vertex v_{**}, uniform in $V(Q_n)$.

6. The metric space $\left(V(Q_n), \left(\frac{9}{8n}\right)^{1/4} d_{Q_n}\right)$

6.1. **Tightness in the Gromov–Hausdorff topology.** Let $(\mathbf{q}, v_*) \in \mathbf{Q}_n^\bullet$ be coded by $(\mathbf{t}, \ell) \in \mathbb{T}_n$ via the CVS bijection, so that $V(\mathbf{t}) = V(\mathbf{q}) \setminus \{v_*\}$. Let $\check{L}(i \to j)$ denote the minimal label along the cyclic interval from e_i to e_j around $\mathbf{t}$.

Lemma 99. $d_{\mathbf{q}}(u_i, u_j) \le L(i) + L(j) - 2\check{L}(i \to j) + 2$

Proof. Look at the arcs $e_i \to s(e_i) \to s^2(e_i) \to \cdots$, and let $e' = s(e)$ be the first corner visited which lies outside the cyclic interval $[e_i, e_j]$. Then $\ell(e) = \ell(e') + 1 = \check{L}(i \to j)$: Also, the chain from e_i^- to $(e')^-$ has length $\ell(e_i) - \ell(e') = L(i) - \check{L}(i \to j) + 1$. Now look at the chain $e_j \to s(e_j) \to s^2(e_j) \to \cdots$. e' must belong here, since it is the first edge after e_i (hence e_j) with label $\check{L}(i \to j) - 1$. The length of the chain from e_j to e' is therefore $\ell(e_j) - \ell(e') = L(j) - \check{L}(i \to j) + 1$. Concatenating the two chains, we obtain the claim. □

Corollary 100. $d_{\mathbf{q}}(u_i, u_j) \le L(i) + L(j) - 2\max\left\{\check{L}(i \to j), \check{L}(j \to i)\right\} + 2$

Remark 101. Note that one of $\left\{\check{L}(i\to j),\check{L}(j\to i)\right\}$ is the minimal label, and so will not intervene. Also, the estimate cannot be precise. In fact, $L(i)+L(j)-2\max\left\{\check{L}(i\to j),\check{L}(j\to i)\right\}$ defines a tree metric, basically the distance d_L. This tree is precisely the tree we saw in the roundabout construction above.

Observation 102. *$d_{\mathbf{q}}(u,v)\geq|\ell(u)-\ell(v)|$, since $|\ell(e)-\ell(s(e))|=1$.*

Exercise 103. $d_{\mathbf{q}}(u,v)\geq\ell(u)+\ell(v)-2\min_{w\in[u,v]_{\mathbf{t}}}\ell(w)$ where $[u,v]_{\mathbf{t}}$ is the unique geodesic in $\mathbf{t}$ connecting u,v.

We now couple the different metric spaces under consideration by setting $\mathcal{D}_{(n)}(i,j)=d_{Q_n}(u_i,u_j)$ for $0\leq i,j\leq 2n$, and adding an extra point $\partial\notin\mathbb{Z}$ for which $\mathcal{D}_{(n)}(\delta,i)=d_{Q_n}(v_*,u_i)$. Then $\left(\{0,1,\cdots,2n,\partial\},\mathcal{D}_{(n)}\right)$ is a pseudometric space and its natural quotient is isometric to $(V(Q_n),d_{Q_n})$. Extend $\mathcal{D}_{(n)}$ to $[0,2n]^2$ bilinearly, in other words set:

(6.1)
$$\begin{aligned}\mathcal{D}_{(n)}(x,y)=(1-\{x\})(1-\{y\})\mathcal{D}_{(n)}\left(\lfloor x\rfloor,\lfloor y\rfloor\right)+(1-\{x\})\{y\}\mathcal{D}_{(n)}\left(\lfloor x\rfloor,\lceil y\rceil\right)\\+\{x\}(1-\{y\})\mathcal{D}_{(n)}\left(\lceil x\rceil,\lfloor y\rfloor\right)+\{x\}\{y\}\mathcal{D}_{(n)}\left(\lceil x\rceil,\lceil y\rceil\right).\end{aligned}$$

Lemma 104. *$\mathcal{D}_{(n)}\colon[0,2n]^2\to\mathbb{R}_{\geq0}$ is continuous and satisfies the triangle inequality. Moreover,*

$$\mathcal{D}_{(n)}(x,y)\leq L(x)+L(y)-2\check{L}(x,y)+2$$

where $L(x)$ is defined by linear interpolation.

Finally, rescale by setting

$$\mathcal{D}_n(s,t)=\left(\frac{9}{8n}\right)^{1/4}\mathcal{D}_{(n)}(2ns,snt)$$

Theorem 105. *The sequence $(\mathcal{D}_n)_{n\geq1}$ is tight in $C\left([0,1]^2,\mathbb{R}\right)$ for the uniform topology.*

Proof. Since $\mathcal{D}_n(0,0)=0$, by the Arzela–Ascoli theorem it is enough to estimate the *modulus of continuity* of the functions $\mathcal{D}_n$. Moreover, by the triangle inequality

$$|\mathcal{D}_n(s,t)-\mathcal{D}_n(s',t')|\leq\mathcal{D}_n(s,s')+\mathcal{D}_n(t,t')$$

so

$$\sup_{|s-s'|,|t-t'|\leq\delta}|\mathcal{D}_n(s,t)-\mathcal{D}_n(s',t')|\leq2\sup_{|t-t'|}\mathcal{D}_n(t,t').$$

By the final claim of Lemma 104, we see that (if we order t,t' so that $t\to t'$ is the shorter interval)

$$\begin{aligned}\mathcal{D}_n(t,t')&\leq\left(\frac{9}{8n}\right)^{1/4}\left(L_n(2nt)+L_n(2nt')-\check{L}_n(2nt\to2nt')+2\right)\\&\leq2\omega\left(\frac{L_n(2n\cdot)}{(8n/9)^{1/4}},|t-t'|\right)+\frac{2}{(8n/9)^{1/4}}.\end{aligned}$$

Since $\frac{L_n(2n\cdot)}{(8n/9)^{1/4}}\xrightarrow[n\to\infty]{(\mathrm{d})}Z$ in the uniform topology, their moduli of continuity converge in probability (see [3]): for every $\epsilon,\eta>0$ there is $\delta>0$ so that

$$\limsup_{n\geq1}\mathbb{P}\left(\omega\left(\frac{L_n(2n\cdot)}{(8n/9)^{1/4}},\delta\right)>\epsilon\right)\leq\eta.$$

Therefore, $\forall \epsilon, \eta > 0 \exists \delta > 0$ such that

$$\limsup_{n\geq 1} \mathbb{P}\left(\omega(\mathcal{D}_n, \delta) > \epsilon\right) \leq \eta\,.$$

Now, given $\epsilon > 0$, for every $k \geq 1$ there is $\delta_k > 0$ such that

$$\sup_{n\geq 1} \mathbb{P}\left(\omega(\mathcal{D}_n, \delta_k) > \frac{1}{2^k}\right) \leq \frac{\epsilon}{2^k}\,.$$

For this first apply the previous claim (with $\epsilon = 2^{-k}$) to get a δ that holds for n large

Fixing n and summing over k we find:

$$\mathbb{P}\left(\forall k : \omega(\mathcal{D}_n, \delta_k) \leq \frac{1}{2^k}\right) \geq 1 - \epsilon\,,$$

which is a uniform bound on the modulus of continuity of the $\mathcal{D}_n$. □

6.2. **The Brownian map.** Since $(\mathcal{D}_n)_{n\geq 1}$ is tight, the same is true for the sequence of triples

$$\left(\left(\frac{C_n(2n\cdot)}{\sqrt{2n}}\right), \left(\frac{L_n(2n\cdot)}{(8n/9)^{1/4}}\right), \mathcal{D}_n\right)_{n\geq 1}\,.$$

We can therefore choose a subsequential limit of this triple. More precisely, we can choose a sequence $(n_k)_{k\geq 1}$ along which this triple convergence to a limit $(\mathbf{e}, Z, \mathcal{D})$.

Remark 106. Until further notice we fix the subsequence, and $\xrightarrow[n\to\infty]{(\mathrm{d})}$ will mean convergence in distribution *along this subsequence*. By the Skorokhod Theorem we may assume the convergence holds almost surely.

Notation 107. $s_* = \arg\min_{s\in[0,1]} Z_s$.

Lemma 108. *s_* is uniquely defined almost surely.*

Proposition 109. *Almost surely for every $t \in [0, 1]$,*

$$\begin{aligned} \mathcal{D}(t, s_*) &= Z_t - Z_{s_*} \\ &= Z_t - \min Z\,. \end{aligned}$$

Proof. $\left(\frac{9}{8n}\right)^{1/4} d_{Q_n}(u_i, v_*) = \frac{L_n(i) - \min L_n + 1}{(8n/9)^{1/4}}$. Now choose $i_* = i_*(n)$ so that $u_{i_*}$is the last visited vertex in a geodesic from u_i to v_*. Then $d_{Q_n}(u_i, u_{i_*}) = d_{Q_n}(u_i, v_*) - 1 = L_n(i) - \min L_n$ so $L_n(i_*) = \min L_n$. Passing to a subsequence we may assume $\left\{\frac{i_*(n)}{2n}\right\}_{n\geq 1}$ converges almost surely. The uniqueness of s_* and the convergence of $\frac{L_n(2n\cdot)}{(8n/9)^{1/4}}$ to Z shoes that $\frac{i_*(n)}{2n} \to s_*$ along the subsequence. Now also choose $i = i(n)$ so that $\frac{i(n)}{2n} \to t$. We now get

$$\begin{aligned} \left(\frac{9}{8n}\right)^{1/4} d_{Q_n}(u_i, v_*) &= \left(\frac{9}{8n}\right)^{1/4} \left(\mathcal{D}_{(n)}(i, i_*) + 1\right) \\ &= \left(\frac{9}{8n}\right)^{1/4} \mathcal{D}_n\left(\frac{i}{2n}, \frac{i_*}{2n}\right) + o(1) \\ &\xrightarrow[n\to\infty]{} \mathcal{D}(t, s_*) \end{aligned}$$

while $\frac{L_n(i)}{(8n/9)^{1/4}} \to Z_t$, $\frac{\min L_n - 1}{(8n/9)^{1/4}} \to Z_{s_*}$. □

Definition 110. Let $(\mathcal{S},\mathcal{D}) = ([0,1],\mathcal{D})\,/\,\{\mathcal{D}=0\}$. Write $\pi\colon [0,1]\to\mathcal{D}$ for the quotient map.

Corollary 111. *(Along n_k) We have the following convergence in the Gromov–Hausdorff metric:*

$$\left(V(Q_n), \left(\frac{9}{8n}\right)^{1/4} d_{Q_n}, v_*\right) \xrightarrow[n\to\infty]{(\mathrm{d})} (\mathcal{S},\mathcal{D},s_*)\ .$$

Proof. By the Skorokhod Representation Theorem we may assume that $\mathcal{D}_n \xrightarrow[n\to\infty]{a.s.} \mathcal{D}$. Certainly $\mathcal{D}$ is non-negative on $[0,1]$ and satisfies the triangle inequality there, since the $\mathcal{D}_n$ do. It vanishes on the diagonal since $\mathcal{D}_n(t,t) \le \left(\frac{9}{8n}\right)^{1/4} \xrightarrow[n\to\infty]{} 0$, so it is a pseudometric. To estimate the Gromov–Hausdorff distance we use the correspondence

$$\mathcal{R} = \{(t,\lfloor 2nt\rfloor) \mid t\in[0,1]\} \cup \{(s_*,\partial)\} \subset [0,1]\times\{0,1,\cdots,2n,\partial\}\ .$$

Its distortion is at most

$$\sup_{t,t'\in[0,1]}\left|\mathcal{D}(t,t') - \left(\frac{9}{8n}\right)^{1/4}\mathcal{D}_{(n)}\left(\lfloor 2nt\rfloor, \lfloor 2nt'\rfloor\right)\right| + \sup_{t\in[0,1]}\left|\mathcal{D}(t,s^*) - \left(\frac{9}{8n}\right)^{1/4}\mathcal{D}_{(n)}\left(\lfloor 2nt\rfloor,\partial\right)\right|\ .$$

The first term is

$$\sup_{t,t'\in[0,1]}\left|\mathcal{D}(t,t') - \left(\frac{9}{8n}\right)^{1/4}\mathcal{D}_n\left(\frac{\lfloor 2nt\rfloor}{2n},\frac{\lfloor 2nt'\rfloor}{2n}\right)\right| \xrightarrow[n\to\infty]{} 0$$

since $\mathcal{D}_n \to \mathcal{D}$ in the uniform topology, and we have an a-priori bound on the modulus of continuity of $\mathcal{D}$.

For the second term,

$$\begin{aligned}
\left(\frac{9}{8n}\right)^{1/4}\mathcal{D}_{(n)}\left(\lfloor 2nt\rfloor,\partial\right) &= \left(\frac{9}{8n}\right)^{1/4} d_{Q_n}\left(u_{\lfloor 2nt\rfloor}, v_*\right)\\
&= \frac{L_n(\lfloor 2nt\rfloor) - \min L_n + 1}{(8n/9)^{1/4}}\\
&\to Z_t - \min Z = Z_t - Z_{s_*}\\
&= \mathcal{D}(t,s_*)
\end{aligned}$$

for the coupling defined above. ☐

Definition 112. The pair $(\mathcal{S},\mathcal{D})$ is called a *Brownian map.*

6.3. Basic Properties of the Brownian map.

- $\mathcal{D}(x,x_*) = Z_x - \min Z$ where $x=\pi(s)$, $x_*=\pi(s_*)$ (note that almost surely for all s,t if $\mathcal{D}(s,t)=0$ then $Z_s=Z_t$, so the notation $Z_{\pi(s)}$ for Z_s makes sense)
- Define $d_Z(s,t) = Z_s + Z_t - 2\max\left(\check{Z}_{s\to t},\check{Z}_{t\to s}\right)$ where the notation refers to the cyclic order on $[0,1]$. Then the natural quotient of $([0,1],d_Z)$ is an $\mathbb{R}$-tree T_Z. It is isometric to the R-tree $T_{\tilde{Z}}$ where $\tilde{Z} = Z_{s+s_*} - Z_{s_*}$ where the addition is mod 1.

Proposition 113. *Almost surely, for all $s,t\in[0,1]$ $\mathcal{D}(s,t)\le d_Z(s,t)$.*

Proof. Immediate from $d_{Q_n}(u_i,u_j) \le L_n(i)+L_n(j)-2\max\left(\check{L}(i\to j),\check{L}(j\to i)\right)+2$. □

Remark 114. So far we have concentrated on $\mathcal{D}$, a function given by the labels, which encode distances in the map. But we also have the tree coded by $\mathbf{e}$, which parametrizes the vertices of the map.

Lemma 115. *Suppose $d_{\mathbf{e}}(s,t)=0$. We can the find $0\le i_n, j_n \le 2n$ for each n so that $\frac{i_n}{2n}\to s$, $\frac{j_n}{2n}\to t$ and $u_{i_n}=u_{j_n}$.*

Proof. [Omitted] □

Proposition 116. *Almost surely, for every $s,t\in[0,1]$ if $d_{\mathbf{e}}(s,t)=0$ then $\mathcal{D}(s,t)=0$.*

Proof. Choose i_n, j_n as in the lemma. Then $\mathcal{D}_n\left(\frac{i_n}{2n},\frac{j_n}{2n}\right)=\mathcal{D}_{(n)}(i_n,j_n)=d_{Q_n}(i_n,j_n)=0$. By the uniform continuity of the $\mathcal{D}_n$ we can pass to the limit and get $\mathcal{D}_n(s,t)=0$. □

Corollary 117. *The relation $\mathcal{D}(s,t)=0$ is a refinement of the transitive closure of the union of $d_{\mathbf{e}}(s,t)=0$ and $d_Z(s,t)=0$.*

Proof. Combine Propositions 113 and 116. □

Claim 118. In fact, we have equality here.

Consider now the set of all pseudometrics d on $[0,1]$ such that $d\le d_Z$ and $d_{\mathbf{e}}(s,t)=0 \Rightarrow d(s,t)=0$. For such d let s,t be fixed, and consider a finite sequence $s=s_1,t_1,s_2,t_2,\ldots,s_k,t_k=t$ such that $d_{\mathbf{e}}(t_i,s_{i+1})=0$ for $1\le i\le k-1$. Then

$$\begin{aligned} d(s,t) &\le \sum_{i=1}^{k} d(s_i,t_i)+\sum_{i=1}^{k-1} d(t_i,s_{i+1}) \\ &\le \sum_{i=1}^{k} d_Z(s_i,t_i)+0\,. \end{aligned}$$

It follows that $d(s,t)\le \mathcal{D}^*(s,t)$ for the pseudometric

$$\mathcal{D}^*(s,t) \stackrel{\text{def}}{=} \inf\left\{\sum_{i=1}^{k} d_Z(s_i,t_i) \mid \{s_i\}_{i=1}^{k},\{t_i\}_{i=1}^{k}\subset[0,1] \text{ with } s_1=s,\ t_k=t\,, d_{\mathbf{e}}(t_i,s_{i+1})=0\right\}.$$

It is easy to check that this is a pseudometric (analogue of the "path metric" construction). Since $\mathcal{D}^*(s,t)\le d_Z(s,t)$ and $d_{\mathbf{e}}(s,t)=0\Rightarrow \mathcal{D}^*(s,t)=0$, $\mathcal{D}^*$ is the maximal element of the class, and in particular $\mathcal{D}\le\mathcal{D}^*\le d_Z$.

Theorem 119 (Main Theorem of these lectures; Le Gall 2011, Miermont 2011)**.** *Almost surely, $\mathcal{D}=\mathcal{D}^*$.*

Corollary 120. *We have almost sure convergence <u>along the full sequence</u>*

$$\left(V(Q_n),\left(\frac{9}{8n}\right)^{1/4} d_{Q_n}, v_*\right) \xrightarrow[n\to\infty]{\text{(a.s.)}} (\mathcal{S},\mathcal{D}^*,x_*)$$

in the Gromov–Hausdorff topology.

Proof of Corollary. The metric $\mathcal{D}^*$ only depends on $\mathbf{e}, Z$ and hence so does $[0,1] \big/ \{\mathcal{D}^* = 0\} = [0,1] \big/ \{\mathcal{D} = 0\} = \mathcal{S}$. It follows that $(\mathcal{S}, \mathcal{D}^*, x_*)$ is equal to every subsequential limit, so the whole sequence converges. □

Note first that for every $x \in \mathcal{S}$ we have $\mathcal{D}(x, x_*) = Z_x - Z_{x_*} = d_Z(s, s_*)$ where $\pi(s) = x$ and $\pi(s_*) = x_*$. Let p_Z be the quotient map $[0,1] \to T_Z = [0,1]/\{d_Z = 0\}$. Then:

Proposition 121. *The identity from* $[0,1]$ *to itself induces a continuous map* $\phi\colon T_Z \to \mathcal{S}$. *Furthermore,* $\phi_* [p_Z(s_*), p_Z(s)]_{T_Z}$ *is a geodesic in* $(\mathcal{S}, \mathcal{D})$ *from* x_* *to* $x = \pi(s)$.

Proof. For any $0 \le r \le r' \le d_Z(s, s_*) = \mathcal{D}(s, s_*)$ let $p_Z(t), p_Z(t')$ be the points at d_Z-distance r, r' from $p_Z(s_*)$ along the geodesic. Then

$$|r - r'| = |\mathcal{D}(\pi(t), x_*) - \mathcal{D}(\pi(t'), x_*)| \le \mathcal{D}(\phi(p_Z(t)), \phi(p_Z(t'))) \le d_Z(t, t') = r' - r\,,$$

so the restriction of ϕ to the geodesic is an isometry. □

6.4. **The volume measure.** The projection π is Hölder continuous of any order $\alpha < \frac{1}{4}$ since $\mathcal{D}(\pi(s), \pi(t)) \le d_Z(s,t) \le 2\|Z\|_{C^{0,\alpha}} |t - s|^\alpha$ where $\|Z\|_{C^{0,\alpha}} < \infty$ almost surely (which gives an alternate proof that $(\mathcal{S}, \mathcal{D})$ is compact). An immediate corolla of the Hölder continuity is the following bound:

Proposition 122. $\dim_H(\mathcal{S}, \mathcal{D}) \le 4$ *almost surely.*

In fact, we can say more. For this m be Lebesgue measure on $[0,1]$, $\lambda = \pi_* m$, a Borel measure on $\mathcal{S}$. It is then enough to show the following:

Proposition 123. *Almost surely* $\forall \epsilon > 0 \exists$*random* $c, \eta > 0$ *such that for every* $x \in S$, $r \in (0, \eta)$

$$\lambda(B_{\mathcal{D}}(x, r)) \ge c r^{4+\epsilon}\,.$$

Proof. For any s such that $\pi(s) = x$,

$$\begin{aligned} \lambda(B_{\mathcal{D}}(x,r)) &= m(\{t \mid \mathcal{D}(s,t) \le r\}) \\ &\ge m(\{t \mid d_Z(s,t) \le r\}) \\ &\ge 2h \end{aligned}$$

where h is such that $2\|Z\|_{C^{0,\alpha}} h^\alpha \le r$. Choosing $\alpha = \frac{1}{4+\epsilon}$ gives the claim. □

Theorem 124 (Le Gall)**.** *Almost surely* $\forall \epsilon \in (0,4)$, *there exist a random* $C \in (0, \infty)$ *such that for all* $x \in cS$, $r > 0$

$$\lambda(B_{\mathcal{D}}(x, r)) \le C r^{4-\epsilon}\,.$$

Corollary 125. $\dim_H(\mathcal{S}, \mathcal{D}) \ge 4$ *almost surely.*

And we obtain equality.

Proposition 126 (Re-rooting the Brownian map)**.** *Let* $(\mathcal{S}, \mathcal{D}, x_*)$ *be "the" Brownian map, and let* $x_{**} \in \mathcal{S}$ *be chosen with law* λ, *conditionally given* $(\mathbf{e}, Z, \mathcal{D})$ *(in practice,* $x_{**} = \pi(U)$ *where* U *is uniform in* $[0,1]$ *and independent of* $(\mathbf{e}, Z, \mathcal{D})$*). Then*

$$(\mathcal{S}, \mathcal{D}, x_*) \overset{(d)}{=} (\mathcal{S}, \mathcal{D}, x_{**})\,.$$

Proof. Recall that $(V(Q_n), d_{Q_n}, v_*) \overset{(\mathrm{d})}{=} (V(Q_n), d_{Q_n}, v_{**})$ where v_{**} is uniform in Vrq. We now replace v_{**} with $u_{**} = u_{\langle U \rangle_n}$ where U is uniform in $[0,1]$. Then u_{**} and v_{**} do not precisely have the same law, but they can be coupled with error that goes to 0 as $n \to \infty$. Then

$$\left(V(Q_n), \left(\frac{9}{8n}\right)^{1/4} d_{Q_n}, u_{**}\right) \xrightarrow[n\to\infty]{\text{(a.s.)}} (\mathcal{S}, \mathcal{D}, x_{**})$$

where $x_{**} = \pi(U)$. To show this, we introduce the correspondence $\mathcal{R}_n = \left\{\left(u_{\lfloor 2nt \rfloor}, \pi(t)\right)\right\}_{t\in[0,1]} \cup \left\{(v_*, x_*), \left(u_{\langle U \rangle_n}, \pi(U)\right)\right\}$. Checking that the distortion of $\mathcal{R}_n$ tends to zero with n almost surely is left as an exercise. $\square$

We will need the following generalization

Proposition 127. *If $\{x_i\}_{i=1}^k$ are iid (λ) in $\mathcal{S}$ then $(\mathcal{S}, \mathcal{D}, x_*, x_2, \ldots, x_k) \overset{(d)}{=} (\mathcal{S}, \mathcal{D}, x_1, x_2, \ldots, x_k)$ in the sense of k-pointed spaces, with the Gromov–Hausdorff metric $d_{GH}\left((X, d_X, x_1, \ldots, x_k), (Y, d_Y, y_1, \ldots, y_k)\right) = \frac{1}{2}\inf\{\mathrm{dist}(\mathcal{R}) \mid \mathcal{R} \in Corr(X, Y), \forall i : (x_i, y_i) \in \mathcal{R}\}$.*

References

[1] David Aldous. The continuum random tree. I. *Ann. Probab.*, 19(1):1–28, 1991.

[2] David Aldous. The continuum random tree. III. *Ann. Probab.*, 21(1):248–289, 1993.

[3] Patrick Billingsley. *Convergence of probability measures.* Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[4] J. Bouttier, P. Di Francesco, and E. Guitter. Planar maps as labeled mobiles. *Electron. J. Combin.*, 11(1):Research Paper 69, 27, 2004.

[5] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[6] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[7] Philippe Chassaing and Gilles Schaeffer. Random planar lattices and integrated superBrownian excursion. *Probab. Theory Related Fields*, 128(2):161–212, 2004.

[8] Ian Chiswell. *Introduction to Λ-trees.* World Scientific Publishing Co. Inc., River Edge, NJ, 2001.

[9] Jean-François Le Gall. Random trees and applications. *Probab. Surv.*, 2:245–311, 2005.