Lecture of 2012-06-08 **scribe: Hannah Cairns**

1 The limit shape of the height function

We know that the TASEP is the same as last passage percolation, and it can also be represented in terms of the corner growth process. We also know that the last passage time converges to a limit, that is, that

$$\frac{1}{n}G(nx, ny) \to g(x, y) \text{ a.s.}$$

where $g(x, y)$ is homogenous and continuous in $(0, \infty)^2$.

We now want to look at the shape of the height function at a certain time, based on our information about $G(nx, ny)$. Let S_t be the set of points that have been reached by time t, using Russian coordinates.

$$S_t = \{(x, y) : G(x, y) < t\}$$

Let h_t be the height function of the TASEP in Russian coordinates:

$$h_t(x) = \max\{y : (x, y) \in S_t\}$$

If we know approximately what time every point is added to the corner process, then we know the limit shape of the height function. Suppose that we have the following convergence property: for every small constant $c > 0$,

$$\sum_{x,y} \mathbb{P}\big[|G(x, y) - g(x, y)| > cg(x, y)\big] < \infty.$$

Then, by Borel-Cantelli, for all but finitely many squares $(x,\ y)$, it will be true that

$$(1 - c)g(x, y) \le G(x, y) \le (1 + c)g(x, y). \tag{1}$$

The exceptional squares will eventually be filled, and they will eventually fall inside the rescaled limit shape as long as $g(x, y) > 0$ everywhere.

So for large enough t, (1) implies

$$\{(x, y) : (1 + c)g(x, y) < t\} \subset S_t \subset \{(x, y) : (1 - c)g(x, y) < t\}.$$

Then $\{(x, y) : g(x, y) < 1 - \varepsilon\} \subset S_t/t \subset \{(x, y) : g(x, y) < 1 + \varepsilon\}$ eventually for every ε, and so, with probability 1, $(1/n)S_t$ converges to the limit shape $\{(x, y) : g(x, y) < 1\}$.

We have a monotonicity property which gives us lower bounds on the shape of $h(x)$. We claim that these lower bounds describe the limit shape. To show this, we are going to go back to the last-passage formulation and prove that

$$h(x) = \begin{cases} \frac{1}{2} + \frac{1}{2}x^2 & \text{if } |x| < 1 \\ |x| & \text{otherwise,} \end{cases}$$

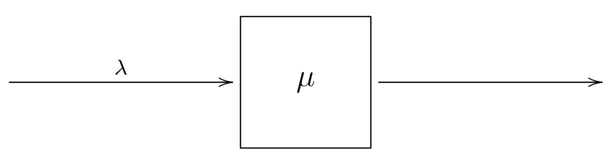
or in Russian coordinates $g(x) = (\sqrt{x_1} + \sqrt{x_2})^2$,

Let $U(\vec{x}) = G(\vec{x}) - G(\vec{x} + \vec{e}_1), V(\vec{x}) = G(\vec{x}) - G(\vec{x} + \vec{e}_2)$, the increments along horizontal and vertical edges. We know that the increments are not all independent, since $U(\vec{x}) + V(\vec{x} - \vec{e}_2) = U(\vec{x} - \vec{e}_1) + V(\vec{x})$, but we can hope to find a stationary, independent distribution for the $U(\vec{x})$ or $V(\vec{x})$ alone.

To do this, we cast the TASEP into one final representation in terms of queues in tandem.

2 Queues in tandem

We state the definition of the $M/M/1$ queue. The 'M' in the first place means that arrivals are 'Markovian,' that is, that they are a Poisson process with some rate λ. and the 'M' in the second place means that the service times are 'Markovian' also, with some Poisson rate μ.



Arrivals enter into a queue, and at every service time, if there is anyone in the queue, one person is served and departs. If there is no one in the queue, nothing happens. This gives us a new process, the process D of departure times, which is determined by the arrival process A and service process S.

Then we recall that:

- if $\lambda < \mu$, the queue stays at a finite size and there is a limiting distribution for the number of people in the queue.
- if $\lambda > \mu$, then the queue becomes infinitely long and there is no stationary distribution.

- if $\lambda = \mu$, then the queue is critically unstable and will become infinitely long as time goes on. There will be on the order of $\sqrt{t}$ people in the queue at time t.

If $\lambda < \mu$, then the number of people in the queue will be distributed like $\text{Geom}(\lambda/(\mu - \lambda))$.

Suppose you have an arrival process and a departure process. How many people are in the queue at any particular time?

The approach is to compare the number of departures in an interval $[s, t]$ from the number of arrivals. Let $Q(t)$ be the length of the queue at time t, let $A(s, t)$ be the number of arrivals in $(s, t]$, let $S(s, t)$ be the number of services in $(s, t]$, and let $D(s, t)$ be the number of departures in $(s, t]$; then

$$\begin{aligned} Q(t) &= Q(s) + A(s,t) - D(s,t) \\ &\geq Q(s) + A(s,t) - S(s,t) \\ &\geq A(s,t) - S(s,t) \end{aligned}$$

so $Q(t) \geq \max\{A(s,t) - S(s,t) \mid s \leq t\}$.

In fact, we have equality here. The reason is that if we choose

$$s_0 = \sup\{s \leq t \mid Q(s) = 0\},$$

then $Q(s_0-) = 0$, and the queue was never empty in $(s_0, t]$, so all of the service times in $(s_0, t]$ resulted in a departure.

Then

$$\begin{aligned} Q(t) &= \lim_{s \to s_0-} Q(s) + A(s,t) - D(s,t) \\ &= A(s_0, t) - D(s_0, t) \end{aligned}$$

almost surely, because the probability of an arrival at any time is zero.

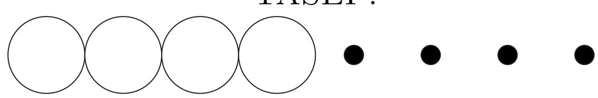
3 TASEP is the same as queues in tandem

We have one last representation of the TASEP in terms of an infinite series of queues.

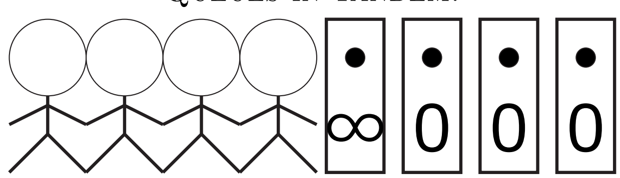
We think of the holes in the TASEP as queues which are served with rate 1, and the particles to the immediate left of a hole as the people waiting at that queue.

Our usual starting position for the TASEP from the queue perspective looks like infinitely many queues, the first one with infinitely many people waiting at it, and the rest initially empty. Each person has to wait at each of the queues in order one by one. When they get out of one queue, they go to the next queue. This process is called “queues in tandem.”[*]

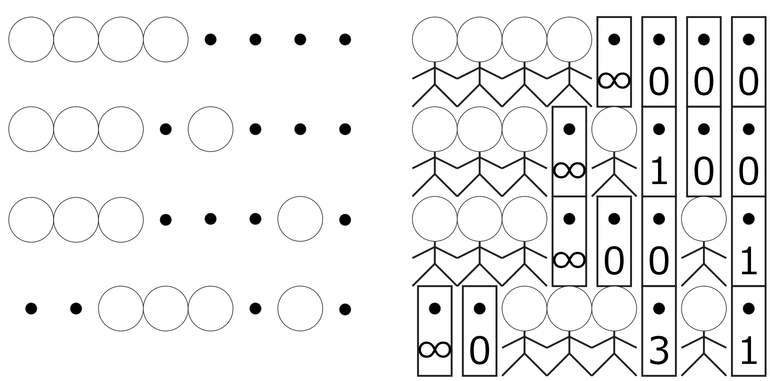
TASEP.



Queues in tandem.



The correspondence in operation.



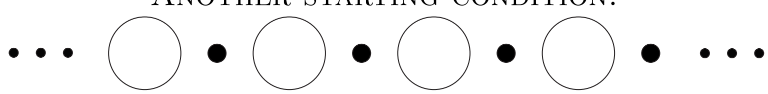
[*]Or sometimes “England.”

In the figure above, the number in each of the queues is the number of people (particles) to the left of it.

In the TASEP, every hole switches with the particle before it with rate 1, unless there is no particle before it. In the representation in terms of tandem queues, every queue passes people through at rate 1, unless there is no one waiting. It is clear that the TASEP and queues in tandem can be coupled together, by choosing the service times of the queue at position $n+1$ to be the same as the swapping times of the pair $(n, n+1)$.

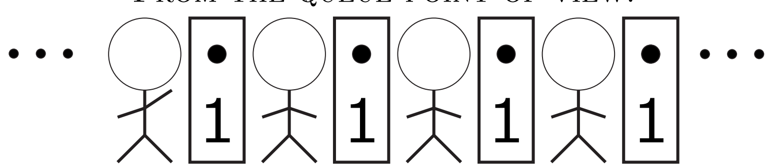
We could change the starting condition. For example, the starting condition for the TASEP that has particles in positions $\ldots, -2, 0, 2, 4, \ldots$,

ANOTHER STARTING CONDITION.



looks from the queue point of view like this:

FROM THE QUEUE POINT OF VIEW.



Or we could have $\text{Geom}(\bar{p})$ people at each queue.

4 The stationary measure for a chain of queues

Our queues serve people at times given by a Poisson process with rate 1, but the arrival process is general. If we start with some distribution on the number of people in the queues and then run the process for some time, we will get a new distribution.

What are the stationary distributions for this process? It turns out that, if they are also translation-invariant, then the only possibilities are $\mu_p = \text{Geom}(\bar{p})^{\mathbb{Z}}$. We will prove that in this section.

We start with the observation that the chain of queues is Markov in the sense that the behaviour of a particular queue only depends on its service process and the departure process of the queue immediately previous to it. The latter process tells us the arrival times of our queue.

So we are given an arrival process, and we produce a departure process.

BURKE'S THEOREM. *Provided that we allow the processes to run for infinite negative time as well as infinite positive time, if the arrival process is a Poisson process* Pois(λ)*, with* $\lambda \leq 1$*, then the output process will also be* Pois(λ)*.*

DIGRESSION:

> How can the process even be defined for negative time, since we need to know how many people are in the queue? We use the formula earlier as a definition:
>
> $$Q(t) = \max\{A(s,t) - S(s,t) \mid s \leq t\}.$$
>
> Then we have a departure whenever there is a service and $Q(t) > 0$.
>
> Here $Q(t)$ will be infinite for every time t if $\lambda = 1$, since the unbiased random walk gets arbitrarily large with probability one, so the output process will just be the service process. So this case is trivial. From now on we take $\lambda < 1$.

If $A \sim$ Pois(λ) with $\lambda < 1$, this difference is a continuous time random walk with a negative drift, since $\lambda < 1$, so it will go to $-\infty$, and $Q(t)$ will be finite. This makes sense, because with an arrival rate lower than the service rate, the queue should empty once in a while.

PROOF: We prove this theorem by writing the arrival and service processes in terms of reversible processes, and then we realize that when we reverse the process, the arrival times and departure times just switch.

The queue Q is $M/M/1$ and has been running for infinite time. It is therefore at the stationary distribution and therefore reversible, so the random process $\bar{Q}(t) = Q(-t+)$, by which we mean the limit from above of Q at $-t$, has the same distribution as Q.

The process of service times that occur when the queue is empty is also reversible. To see this, just let R be the process which matches S whenever $Q = 0$ (that is, if the queue is empty, the events in R are just the service times in S), and is an independent Pois(1) random process on the set $Q > 0$.

This is really an independent Pois(1) process, since the service times when $Q = 0$ don't affect the value of Q. So it is clearly reversible, and (Q, R) has the exact same distribution as $(\bar{Q}, \bar{R})$.

We have represented the process in terms of reversible processes.

We can recover the arrival and service processes if we know (Q, R), since arrival shows up as increases in Q, and services show up as decreases in Q or event times in R. But if we reverse (Q, R) and try to reconstruct the arrival and service processes with $(\bar{Q}, \bar{R})$, the processes that we recover are different. Arrivals in A show up as increases in Q, which are decreases in $\bar{Q}$, which appear as departures in the reversed process. Departures show up as decreases in Q, which appear as arrivals in the reversed process. Events in R still correspond to service times when the queue is empty or nothing in particular when the queue has things in it.

Since (Q, R) has the same distribution as $(\bar{Q}, \bar{R})$, (A, D) has the same distribution as $(\bar{D}, \bar{A})$. It follows that the reversed departure process is $\mathrm{Pois}(\lambda)$, and because this is reversible so is the original departure process.

This proves the result.

5 Stationary measures for last-passage percolation.

We go back to the last-passage percolation formulation of the TASEP. Recall that the last passage time $G(\vec{x})$ was the amount of time it took to visit the square $\vec{x}$ from 0. In the TASEP, $G(i, \ell)$ is the time it takes for particle i to switch with hole ℓ.

Suppose $G(i, \ell - 1)$ is known, and I want to find out $G(k, \ell)$. Any path to the square (j, ℓ) can be broken up at its last vertical step, and it becomes a path to some square $(i, \ell - 1)$, a vertical step to (i, ℓ), and then horizontal movement to (k, ℓ). So

$$G(k, \ell) = \max_{i \leq k} \left\{ \sum_{j=i}^{k} X_{(j,\ell)} + G(i, \ell - 1) \right\}.$$

And $G(i, \ell)$ are exactly the service times for the ℓ-th queue.

This is sort of like our last problem in that we have a series of random variables and we're coming up with a new series. There is a stationary measure for this problem too, provided that we change the process a little.

We said before that $G(i, 0) = G(0, j) = 0$, that all the squares along the left and bottom edges were already filled, but now we are going to let them be filled randomly with independent increments.

Remember $U(i,j) = G(i,j) - G(i,j-1)$ and similarly there are vertical increments $V(i,j) = G(i,j) - G(i-1,j)$.

Our boundary conditions will be that the horizontal increments along the first row are i.i.d. with distribution $U(i,0) \stackrel{D}{=} \mathrm{Exp}(\alpha)$, and that the vertical increments on the first column are i.i.d. with distribution $V(0,j) \stackrel{D}{=} \mathrm{Exp}(\bar{\alpha})$, where we define as before $\alpha = 1 - \bar{\alpha}$.

So the horizontal increments along the row $(0,j)$ are independent with distribution $\mathrm{Exp}(\alpha)$; we claim that the horizontal increments along every row (i,j) are independent with the same distribution! Of course, they aren't independent of the increments in other rows. We also claim that the vertical increments along every column are independent with distribution $\mathrm{Exp}(\bar{\alpha})$.

This is true for every $\bar{\alpha}$.

I want to show you a direct proof of this claim, although you can get it indirectly from Burke's theorem.

Suppose that we define U', V', U, V as follows.

Increments.

$$
\begin{tikzcd}
\boxed{(i,j-1)} \arrow[d, "V"'] & \boxed{(i,j)} \arrow[l, "U'"'] \arrow[d, "V'"] \\
\boxed{(i-1,j-1)} & \boxed{(i-1,j)} \arrow[l, "U"']
\end{tikzcd}
$$

Let $X = X_{(i,j)}$. We assume U, V, X are independent with the respective distributions $\mathrm{Exp}(\alpha), \mathrm{Exp}(\bar{\alpha}), \mathrm{Exp}(1)$.

Here $G(i,j) = G(i,j-1) \vee G(i-1,j) + X = G(i-1,j-1) + U \vee V + X$. We have the relations

$$
\begin{aligned}
U' &= G(i,j) - G(i,j-1) \\
&= [G(i-1,j-1) + U \vee V + X] - [G(i-1,j-1) + V] \\
&= U \vee V + X - V
\end{aligned}
$$

and $V' = G(i,j) - G(i-1,j) = U \vee V + X - U$.

Lemma. *$U', V', U \wedge V$ have the same joint distribution as U, V, X.*

Proof. First, $U' - V' = U - V$, which is immediate from the definitions. And it is not difficult to see that $U' = X + (U - V)^+$ and $V' = X + (V - U)^+$.

We need a property of the exponential. If U, V are independent variables with distributions $\mathrm{Exp}(\lambda_1), \mathrm{Exp}(\lambda_2)$, then $U - V$ is independent of $U \wedge V$. So U' and V' are also independent of $U \wedge V$, because they are deterministic functions of X and $U - V$.

Not only that, but $U \wedge V$ has the same marginal distribution as X, since it's the minimum of $U \sim \mathrm{Exp}(\alpha)$ and $V \sim \mathrm{Exp}(\bar{\alpha})$. So we can just switch out $U \wedge V$ for X.

$$\begin{aligned} U' &= X &&+ (U - V)^+ \qquad & V' &= X &&+ (V - U)^+ \\ U &= U \wedge V &&+ (U - V)^+ \qquad & V &= U \wedge V &&+ (V - U)^+ \end{aligned}$$

In each sum, the summands are independent random variables, so U', V' clearly have the same joint distribution as U, V.

It follows immediately from the independence of $U \wedge V$ from U' and V' that $U', V', U \wedge V$ have the same joint distribution as U, V, X.

We now prove the property of the exponential that we needed.

Lemma. *If $U \sim \mathrm{Exp}(\alpha)$, $V \sim \mathrm{Exp}(\bar{\alpha})$, then $U \wedge V$ is independent of $U - V$.*

Proof. The joint characteristic function is

$$\int_{x,y \geq 0} (\alpha\bar{\alpha}) e^{-\alpha x - \bar{\alpha} y - is(x \wedge y) - it(x-y)}\, dx\, dy.$$

Make the variable substitution $\mu = x \wedge y$, $\delta = x - y$, which is bijective and which has Jacobian ± 1.

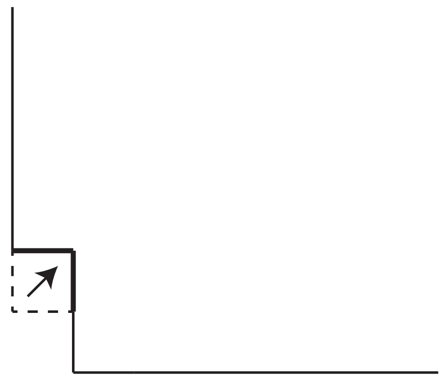
This turns the integral above into

$$\int_{\mu,\delta : \mu \geq 0} (\alpha\bar{\alpha}) e^{(-1-is)\mu + (-\alpha - it)\delta^+ + (-\alpha + it)\delta^-}\, d\mu\, d\delta = f(s) g(t).$$

So $U \wedge V \perp U - V$.

Now that we have this lemma, we can use it over and over to get the main theorem. Suppose we start with a path in the quarter-plane which only moves down and right. Our induction assumption will be that we know that all of the increments along this path are independent, and are $\mathrm{Exp}(\alpha)$ if they're horizontal or $\mathrm{Exp}(\bar{\alpha})$ if they're vertical.

Filling in a corner.



If we fill in a corner on the path, then we throw away the increments U and V and add the increments U' and V' to the path. These new increments are $\text{Exp}(\alpha)$ and $\text{Exp}(\bar{\alpha})$ random variables which are independent of each other, and they are independent of every other increment on the path by the assumption.

So our new path also has the property that all the increments are independent, and that horizontal increments are $\text{Exp}(\alpha)$ and vertical increments are $\text{Exp}(\bar{\alpha})$.

We start with the infinite path

$$C_0 = \ldots, (3,0), (2,0), (1,0), (0,0), (0,1), (0,2), (0,3)\ldots$$

and fill in corners. The property is preserved every time we add a corner anywhere, and it's true when we start, so it's always true.

We have therefore proved the theorem,

Theorem. *The increments on any path that we can get from the path above by filling in finitely many corners are independent and* $\text{Exp}(\alpha)$ *if they are horizontal, or* $\text{Exp}(\bar{\alpha})$ *if they are vertical.*

This proof only deals with paths that have had finitely many corners filled in, but a special case of this is that any finite number of increments on any row are independent, and that means that they are all independent.

If we use this fact, we can prove a concentration property for the last-passage times with our modified boundary condition.

6 A concentration property for the last-passage times in the modified system

Lemma. *Let G_α be the last-passage percolation time for our modified system. There exist K, c so that we have*

$$\mathbb{P}\left[\left|G_\alpha(x) - \frac{x_1}{\alpha} - \frac{x_2}{1-\alpha}\right| > a\sqrt{x_1} + b\sqrt{x_2}\right] < Ke^{-ca-cb}$$

Proof. By the definition of an increment,

$$G_\alpha(x) = \sum_{k=1}^{x_1} V(k,0) + \sum_{\ell=1}^{x_2} U(x_1,\ell).$$

We have just proved that $U(i,\ell)$ with i fixed and $V(k,j)$ with j fixed are independent random variables with distributions Exp() and Exp() respectively, so each summand is really a sum of independent random variables. Exponential random variables are nice enough to be concentrated around their mean.

The mean of the first sum is $x_1/(\alpha)$, and the mean of the second sum is $x_2/(1-\alpha)$.

The sums are not independent of each other, but they are both concentrated near their mean: $\mathbb{P}(|\sum V(k,0) - x_1/(1-\alpha)| > a\sqrt{x_1}) < Ke^{-ca}$. Adding two concentrated random variables together will not ruin the concentration, so we have proved the claim above.

This tells us that $\sum_{x_1,x_2} \mathbb{P}(|G_\alpha(x) - x_1/(1-\alpha) + x_2/\alpha| > c||x||) < \infty$, so

$$G_\alpha(x) = \frac{x_1}{1-\alpha} + \frac{x_2}{\alpha} + o(||x||).$$

We have proved this for the situation with particular independent increments on the edges, not for our original problem.

We will finish the proof of the shape of last-passage percolation in the next lecture.