On Pólya Urn Scheme with Infinitely Many Colors

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Genaralization of the Polya Urn scheme to infinitely many colors

- We introduce an urn with infinite but countably many colors/types of balls indexed by $\mathbb{Z}$.

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- The intial configuration of the urn U_0 is taken to be a probability vector and can be thought to be the proportion of balls of each color/type we start with. Then $\mathbb{P}\left(\text{A ball of color } j \text{ is selected at the first trial} \mid U_0\right) = U_{0,j}$.
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$$U_{n+1} = U_n + I_{n+1}R \tag{1}$$

where $I_n = (\ldots, I_{n,-1}, I_{n,0}, I_{n,1} \ldots)$ where $I_{n,i} = 1$ for $i = j$ and 0 elsewhere.
We study this process for the replacement matrices R which arise out of the Random Walks on $\mathbb{Z}$.

We can generalize this process to general graphs on $\mathbb{R}^d$, $d \geq 1$. Let $G = (V, E)$ be a connected graph on $\mathbb{R}^d$ with vertex set V which is countably infinite. The edges are taken to be bi-directional and there exists $m \in \mathbb{N}$ such that $d(v) = m$ for every $v \in V$. Let the distribution of X_1 be given by

$$\mathbb{P}\left(X_1 = \mathbf{v}\right) = p(\mathbf{v}) \text{ for } \mathbf{v} \in B \text{ where } \mid B \mid < \infty. \tag{2}$$

where $\sum_{\mathbf{v} \in B} p(\mathbf{v}) = 1$. Let $S_n = \sum_{i=1}^{n} X_i$.

Let R be the matrix/operator corresponding to the random walk S_n and the urn process evolve according to R. In this case, the configuration U_n of the process is a row vector with co-ordinates indexed by V. The dynamics is similar to that in one-dimension, that is an element is drawn at random, its type noted and returned to the urn. If the $\mathbf{v}^{\text{th}}$ type is selected at the $n+1^{\text{st}}$ trial, then

$$U_{n+1} = U_n + e_{\mathbf{v}} R \tag{3}$$

where $e_{\mathbf{v}}$ is a row vector with 1 at the $\mathbf{v}^{\text{th}}$ co-ordinate and zero elsewhere.

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- Hence $\frac{U_n}{n+1}$ is a **random** probability vector. For every $\omega \in \Omega$, we can define a random d-dimensional vector $T_n(\omega)$ with law $\frac{U_n(\omega)}{n+1}$.
- Also $\frac{(\mathbb{E}[U_{n,\mathbf{v}}])_{\mathbf{v} \in V}}{n+1}$ is a probability vector. Therefore we can define a random vector Z_n with law $\frac{(\mathbb{E}[U_{n,\mathbf{v}}])_{\mathbf{v} \in V}}{n+1}$.

Literature is available only for finitely many types/ colors. It is known that the asymptotic behavior of the urn model depends on the qualitative properties (transience or recurrence) of the underlying Markov Chain of the replacement matrix.

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Theorem

Let the process evolve according to a random walk on $\mathbb{R}^d$ with bounded increments. Let the process begin with a single ball of type $\mathbf{0}$. For $X_1 = \left(X_1^{(1)}, X_1^{(2)} \ldots X_1^{(d)}\right)$, let $\mu = \left(\mathbb{E}[X_1^{(1)}], \mathbb{E}[X_1^{(2)}], \ldots \mathbb{E}[X_1^{(d)}]\right)$ and $\Sigma = [\sigma_{ij}]_{d\times d}$ where $\sigma_{i,j} = \mathbb{E}[X_1^{(i)} X_1^{(j)}]$. Let B be such that Σ is positive definite. Then

$$\frac{Z_n - \mu \log n}{\sqrt{\log n}} \xrightarrow{d} N(\mathbf{0}, \Sigma) \text{ as } n \to \infty \tag{4}$$

where $N(\mathbf{0}, \Sigma)$ denotes the d-dimensional Gaussian with mean vector $\mathbf{0}$ and variance-covariance matrix Σ. Furthermore there exists a subsequence $\{n_k\}$ such that as $k \to \infty$ almost surely

$$\frac{T_{n_k} - \mu \log n}{\sqrt{\log n}} \xrightarrow{d} N(\mathbf{0}, \Sigma) \tag{5}$$

Corollary

Let $d \geq 1$ and we consider the SSRW. Let the process begin with a single ball of type $\mathbf{0}$. If Z_n be the random d-dimensional vector corresponding to the probability distribution $\frac{(\mathbb{E}[U_{n,v}])_{v \in \mathbb{Z}^d}}{n+1}$, then

$$\frac{Z_n}{\sqrt{\log n}} \xrightarrow{d} N\left(\mathbf{0}, d^{-1}\mathbb{I}_d\right) \text{ as } n \to \infty \tag{6}$$

where $\mathbb{I}_d$ is the d-dimensional identity matrix. Furthemore, there exists a subsequence $\{n_k\}$ such that almost surely as $k \to \infty$

$$\frac{T_{n_k}}{\sqrt{n_k}} \xrightarrow{d} N\left(\mathbf{0}, d^{-1}\mathbb{I}_d\right). \tag{7}$$

Corollary

Let $d = 1$ *and* $\mathbb{P}(X_1 = 1) = 1$. *Let* $U_0 = 1_{\{0\}}$. *If* Z_n *be the random variable corresponding to the probability mass function* $\frac{\left(\mathbb{E}[U_{n,k}]\right)_{k\in\mathbb{Z}}}{n+1}$, *then*

$$\frac{Z_n - \log n}{\sqrt{\log n}} \xrightarrow{d} N(0,1) \text{ as } n \to \infty. \tag{8}$$

Also there exists a subsequence n_k *such that almost surely as* $k \to \infty$

$$\frac{T_{n_k} - \log n_k}{\sqrt{n_k}} \xrightarrow{d} N(0,1). \tag{9}$$

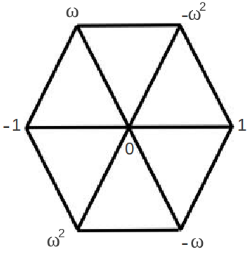

Figure: Triangular Lattice

Corollary

Let the urn model evolve according to the random walk on triangular lattice on $\mathbb{R}^2$ *and the process begin with a single particle of type* 0*, then as* $n \to \infty$

$$\frac{Z_n}{\sqrt{\log n}} \xrightarrow{d} N\left(\mathbf{0}, \frac{1}{2}\mathbb{I}_2\right). \tag{10}$$

Corollary (continued)

Furthermore, there exists a subsequence $\{n_k\}$ such that as $k \to \infty$,

$$\frac{T_{n_k}}{\sqrt{\log n_k}} \xrightarrow{d} N\left(0, \frac{1}{2}\mathbb{I}_2\right) \tag{11}$$

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- On $\mathbb{Z}$, the random walks are recurrent or transient depending on $\mathbb{E}[X_1] = 0$ or not. Asypmtotically both behave similarly upto centering and scaling.
- We conjecture that in the infinite type/ color case, the asymptotic behavior of the process is not determined completely by the underlying Markov Chain of the operator, but by the qualitative properties of the underlying graph.

Proof of the Main Theorem

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- For every $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$, $e(\mathbf{t}) = \frac{1}{4} \sum_{\mathbf{u} \in N(\mathbf{0})} e^{\langle \mathbf{u}, \mathbf{t} \rangle}$ is an eigen value for the operator R where $\mathbf{0}$ stands for the origin in $\mathbb{Z}^2$ and $\langle ., . \rangle$ stands for the inner product.

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- The corresponding right eigen vectors are $\underline{x}(\mathbf{t}) = (x_{\mathbf{t}}(\mathbf{v}))_{\mathbf{v} \in \mathbb{Z}^2}$ where $x_{\mathbf{t}}(\mathbf{v}) = e^{\langle \mathbf{t}, \mathbf{v} \rangle}$.

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- We have noted earlier that $\frac{U_n}{n+1}$ is a **random** probability vector.
- The moment generating function for this vector is given by $\frac{U_n . \underline{x}(\mathbf{t})}{n+1}$ for every $\mathbf{t} \in \mathbb{R}^2$.

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- The moment generating function for this vector is given by $\frac{U_n . \underline{x}(\mathbf{t})}{n+1}$ for every $\mathbf{t} \in \mathbb{R}^2$.
- Using (1), it can be shown that $\overline{M}_n(\mathbf{t}) = \frac{U_n . \underline{x}(\mathbf{t})}{\Pi_n(e(\mathbf{t}))}$ is a non-negative martingale, where $\Pi_n(\beta) = \displaystyle\prod_{j=1}^{n} (1 + \frac{\beta}{j})$.

- Since we begin with one element of type $\mathbf{0}$,

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- We will show that for a suitable $\delta > 0$ and for all $\mathbf{t} \in [-\delta, \delta]^2$

$$\frac{E_n.\,\underline{x}(\frac{\mathbf{t}}{\sqrt{\log n}})}{n+1} \longrightarrow e^{\frac{\|\mathbf{t}\|_2^2}{4}} \tag{13}$$

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- We know that

$$E_n.\,\underline{x}\left(\mathbf{t}_n\right) = \Pi_n\left(e\left(\mathbf{t}_n\right)\right) \tag{14}$$

where $\mathbf{t}_n = \frac{\mathbf{t}}{\sqrt{\log n}}$.

- We use the following fact due to Euler,

$$\frac{1}{\Gamma(\beta+1)} = \lim_{n\to\infty} \frac{\Pi_n(\beta)}{n^\beta}$$

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- It is easy known that this convergence is uniform for all $\beta \in [1-\eta, 1+\eta]$ for a suitable choice of η.
- Due to the uniform convergence, it follows immediately that $\forall \mathbf{t} \in [-\delta, \delta]^2$

$$\lim_{n\to\infty} \frac{\Pi_n\left(e\left(\mathbf{t}_n\right)\right)}{n^{e(\mathbf{t}_n)}/\Gamma(e\left(\mathbf{t}_n\right)+1)} = 1. \tag{15}$$

- Simplifying the left hand side of [13] we get

$$\frac{\Pi_n\left(e\left(\mathbf{t}_n\right)\right)}{n+1} \tag{16}$$

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- Expanding $e\left(\mathbf{t}_n\right)$ into power series and noting that $\Gamma(x)$ is continuous as a function of x we can prove (17).

Thank You!