Lecture 2:

Theorem 1:

$(G_n, x^n) \longrightarrow (G, x)$ in $\mathcal{G}_*[\mathcal{X}]$

implies

$(G_n, X^{G_n, x^n}) \longrightarrow (G, X^{G,x})$ in $\mathcal{G}_*[\mathcal{X}^\infty]$.

Idea of the proof:

- Markov chain case
- $X_v^{G_n, x^n}(k+1) = F\left(X_v^{G_n, x^n}(k), \left(X_u^{G_n, x^n}(k)\right)_{u \sim v}, \xi_v^{(k+1)}\right)$

Inductive argument:

$(G_n, X^{G_n}(0)) \longrightarrow (G, X^G(0))$ by assumption.

Need to prove

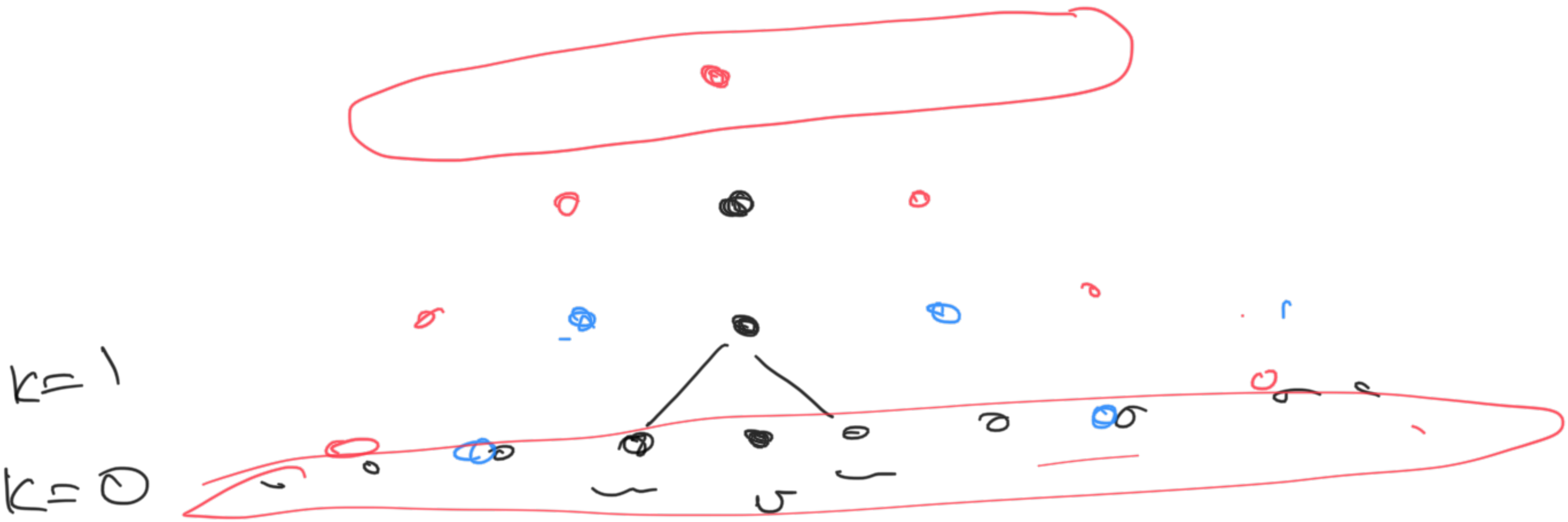
$$\circledast \quad (G_n, X^{G_n}[k]) \longrightarrow (G, X^G[k])$$

where $X[k] = (X(1), \dots, X(k))$.

If you know $\circledast$ holds, then

$$X_v[k+1] = \psi\left((X_u[k])_{u \in B_k(v)}, \xi_v(k+1)\right).$$

for $\psi : \mathcal{X}^k \times \Sigma \longrightarrow \mathcal{X}$



conical dependence

early.

You can prove convergence

Diffusion case $\sigma = I_d$

① $$dX_v^G(t) = b\left(X_v^G(t), \left(X_u^G(t)\right)_{u \sim v}\right)dt + dW_v(t)$$

$$X_v^G(0) = x_v.$$

- If you $\tilde{X}_v^{G,n}$ with the same dynamics different $\{\tilde{x}^n\}$. initial conditions that converge to x, then std. methods show $\tilde{X}^{G,n} \longrightarrow X^G$.

Main steps:

- Sufkces to show $\forall\, T \in (0, \infty)$, $(G, X^{G,x}[T])$.

$(G_n, X^{G_n, x^n}[T]) \longrightarrow (G, \ldots)$

- If $G_n \to G$ and $x_v^n \to x_v$, easy.
- Fix $\varepsilon > 0$, $\ell \in \mathbb{N}$. Fix large $k > \ell$.

Choose $N = N(k)$ s.t. for all $n \geq N(k)$, $\exists$ isomorphism $\varphi_n : B_{k+1}(G) \to B_{k+1}(G_n)$ s.t.

$$\max_{v \in B_{k+1}(G)} |x_v - x^n_{\varphi_n(v)}|^2 \leq \frac{\varepsilon}{3} \quad - (1)$$

- want to estimate distance between the laws of $(X^{G,x})$ and $(X_v^{G_n, x^n})_{v \in G_n}$

$(W_v)_{v \in G}$

- ~~A~~Aim: To couple the processes.

Suppose $\{B_v^n\}_{v \in G_n, n \in \mathbb{N}}$ i.i.d, BMs.

Define

$$W_v^n = \begin{cases} W_{\varphi_n^{-1}(v)}, & v \in B_{k+1}(G_n) \\ B_v^n, & v \in G_n \setminus B_{k+1}(G_n) \end{cases}$$

Define

$$② \quad dX_v^n(t) = b(X_v^n(t), X^n_{N_v(G_n)}(t))dt + dW_v^n(t), \quad X_v^n(0) = x_v^n.$$

Then $X^n \overset{(d)}{=} X^{G_n, x^n}$.

Define $Y_v^n = X^n_{\varphi_n(v)}$, for $v \in B_{k+1}(G_n)$, then $W^n_{\varphi_n(v)} = W_v$.

③ $dY_v^n(t) = b\left(Y_v^n(t), Y^n_{N_v(G_n)}(t)\right)dt + dW_v(t)$

$$Y_v^n(0) = x^n_{\varphi_n(v)}$$

So now we can compare Y^n and X.

by standard methods to show that

$$\Delta_v^n(t) := \mathbb{E}\left[\|X_v - Y_v^n\|^2_{*,t}\right], \quad t \in [0,T]$$

std. inequalities will tell you that

$$\Delta_v^n(t) \leq 3\left|x^n_{\varphi_n(v)} - x_v\right|^2 + \frac{1}{2}C_1 \int_0^t \left(\Delta_v^n(s) + \frac{1}{N_v(G)} \sum_{u \in N_v(G)} \Delta_u^n(s)\right) ds$$

$$\leq \varepsilon + \underbrace{C_1}_{} \int_0^{t} \max_{u:\, d_G(u,v)\leq 1} \Delta_u(s)\, ds.$$

reiterate by applying this same estimate to the vertex u.

iterate $m = k-\ell$ times

$$\leq \varepsilon e^{C_1 T} + \Delta M \frac{(C_1 t)^{k-\ell}}{(k-\ell)!}$$

Choose k s.t. $k-\ell > C_1 T$.

$$\max_{v \in B_\ell(G)} \mathbb{E}\left[\| X^n_{\varphi_n(v)} - X_v \|^2_{*,T} \right]$$

$$C_1 T \qquad \Delta M \, (C_1 T)^{k-\ell}$$

$$\leq \varepsilon e \cdots + \cdots \frac{\cdots}{(k-\ell)!} \Big]$$

Send $n \to \infty$, then $k \to \infty$, $\ell \to \infty$, $\varepsilon \downarrow 0$, to get the result.

QED

$(G_n, x^n) \to (G, x)$ in $\mathcal{G}_*[\mathcal{X}]$

Q2: Empirical measure convergence

$$\mu^{G_n, x^n} := \frac{1}{|G_n|} \sum_{v \in G_n} \delta_{X^{G_n, x^n}}$$

Q2a: Does $\mu^{G_n, x^n} \to$ det limit μ.

If it does, then does $\mu^{G, x} \cdots$

Q2b: If

$$\mu \overset{(d)}{=} \mathrm{Law}(X_g)\,.$$

Definition 1: Consider a sequence of finite graphs G_n and let G be a random $\mathcal{G}_*$-valued element. G_n is said to converge to G in probability in the local weak sense if

$$\boxed{\text{LCP}} \qquad \frac{1}{|G_n|} \sum_{v \in G_n} f(C_v(G_n)) \xrightarrow{(P)} \mathbb{E}[f(G)] \quad \forall f \in \mathcal{C}_b(\mathcal{G}_*).$$

for any graph H ... is the connected

where $C_v(H)$ is component of H containing v.

Definition 2: analogous for marked graphs ...

$$\frac{1}{|G_n|} \sum_{v \in G_n} f(C_v(G_n, y^n)) \xrightarrow{(P)} \mathbb{E}[f(G, y)]$$

$$\forall f \in C_b(\mathcal{G}_*[\mathcal{Y}]).$$

Exercise LCP:

$\boxed{LCP}$ Equivalent to

$$\frac{1}{|G_n|} \sum_{v \in G_n} \delta_{C_v(G_n)} \xrightarrow{(P)} \mathcal{L}aw(G).$$

in $\mathcal{P}(\mathcal{G}_*)$.

$\overline{}$ G in

Definition 3: G_n converges to G in law if

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{1}{|G_n|}\sum_{v\in G_n} f(C_v(G_n))\right] \to \mathbb{E}[f(G)],$$

$f \in C_b(\mathcal{G}_*)$.

Theorem 2: (Lacker-R-Wu, '19).

If (G_n, x^n) converges in probability to a random $\mathcal{G}_*[\mathcal{X}]$-element (G, x).

~~and~~ and $|G_n| \to \infty$, then

$$\mu^{G_n, x_n} \longrightarrow \mathrm{Law}(X_\rho^{G,x}).$$

$$\mu^{G_n,x_n} = \frac{1}{|G_n|}\sum_{v\in G_n} \delta_{X^{G_n,x_n}}$$

<u>Theorem 3</u>:

$G_n := C_\rho(\mathcal{G}(n, \frac{c}{n}))$

$\mu^{G_n,x^n} \longrightarrow$ stoch. limit

Remember $C_\rho(\mathcal{G}(n, \frac{c}{n}))$ weakly local $G = GW(\text{Poisson}(c))$ graph.

whenever $G_n \xrightarrow{\text{local}} G$, G is finite

Note :

$$\mu^{G_n, x^n} \longrightarrow \underbrace{\mu^{G,x}}_{GW(\text{Poisson}(c))} .$$

$GW(\text{Poisson}(c))$ is finite when $c < 1$.

$c > 1$?

$$\mu^{G_n, x^n} \longrightarrow \begin{cases} \text{Law}(X_\rho^\phi) & \text{on } \{\text{no extinction}\} \\ \mu^{G,x} & \{\text{extinction}\} \end{cases}$$