CRM-PIMS SUMMER SCHOOL 2021: BACKGROUND MATERIAL FOR MINI COURSE ON ASYMPTOTICS OF INTERACTING STOCHASTIC PROCESSES ON SPARSE GRAPHS

KAVITA RAMANAN, BROWN UNIVERSITY

1. Local Weak Convergence – Basic Results

The framework we adopt for the convergence analysis is that of *local weak convergence*, a natural mode of convergence for sparse graphs. Essentially, a sequence of (rooted, locally finite) graphs $\{G_n\}_{n\in\mathbb{N}}$ converges locally to a limiting graph G if for each $r > 0$ the neighborhood of radius r around the root G_n is isomorphic to that of G for large enough n; see Section 1.1 for a precise definition. Notably, as summarized in Section 1.3, local limits are well known for many common sparse random graph models: Erdős-Rényi graphs converge to Galton-Watson trees with Poisson offspring distribution, whereas random regular graphs converge to (non-random) infinite regular trees, and configuration models converge to so-called unimodular Galton-Watson trees. The key notion that will be used in the dynamical setting considered here is a similar local convergence notion that can be defined for *marked graphs* (G, y), in which elements $y = (y_v)_{v\in G}$ of some fixed metric space $\mathcal{Y}$ are attached to the vertices of the graph. In our setting, the marks will represent either initial conditions $x = (x_v)_{v\in V}$ or the (random) trajectories $(X_v^{G,x})_{v\in V}$ of the interacting process.

This section describes the basic concepts of local convergence for marked and unmarked graphs. For full details and proofs, see [4, Section 3.2] and Section 2. The notion of local weak convergence was introduced by Benjamini and Schramm in [2]; other useful references on this topic include [1, 4, 10].

1.1. **Unmarked graphs and the space $\mathcal{G}_*$.**

A *rooted graph* $G = (V, E, ø)$ is a graph (V, E) (assumed as usual to be locally finite with either finite or countable vertex set) with a distinguished vertex $ø \in V$. We say two rooted graphs $G_i = (V_i, E_i, ø_i)$ are *isomorphic* if there exists a bijection $\varphi : V_1 \mapsto V_2$ such that $\varphi(ø_1) = ø_2$ and $(\varphi(u), \varphi(v)) \in E_2$ if and only if $(u, v) \in E_1$, for each $u, v \in V_1$. We denote this by $G_1 \cong G_2$. We refer to the map φ as an *isomorphism* from G_1 to G_2, and denote by $I(G_1, G_2)$ the collection of all such isomorphisms from G_1 to G_2.

Let $\mathcal{G}_*$ denote the set of isomorphism classes of connected rooted graphs. Given $k \in \mathbb{N}$ and $G = (V, E, ø) \in \mathcal{G}_*$, let $B_k(G)$ denote the induced subgraph (rooted at ø) consisting of those vertices whose graph distance from ø is no more than k. We say that a sequence $\{G_n\} \subset \mathcal{G}_*$ *converges locally* to $G \in \mathcal{G}_*$ if, for every $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $B_k(G_n) \cong B_k(G)$ for every $n \geq n_k$. There is a metric compatible with this notion of convergence that renders $\mathcal{G}_*$ a complete and separable space, such as

$$d_*(G, G') = \sum_{k=1}^{\infty} 2^{-k} \, 1_{\{I(B_k(G), B_k(G'))=\emptyset\}} \tag{1.1}$$

where as usual $1_{\{A\}} = 1$ if A holds and $1_{\{A\}} = 0$ otherwise.

Remark 1.1. We will often omit the root from the notation, writing $G \in \mathcal{G}_*$ instead of $(G, ø) \in \mathcal{G}_*$, when there is no need to make explicit reference to the root. But we understand that a graph $G \in \mathcal{G}_*$ always carries with it a root, which by default will be denoted ø.

1.2. **Marked graphs and the space $\mathcal{G}_*[\mathcal{Y}]$.** We also need a notion of local convergence for *marked* graphs, where each vertex of the graph has a mark (or label) associated to it; as mentioned, these marks will later encode initial conditions or trajectories of particles. For a metric space $(\mathcal{Y}, d)$, a *$\mathcal{Y}$-marked rooted graph* is a pair (G, y), where $G = (V, E, \emptyset) \in \mathcal{G}_*$, and $y = (y_v)_{v \in V} \in \mathcal{Y}^V$ is a vector of marks. For a $\mathcal{Y}$-marked rooted graph (G, y) and $k \in \mathbb{N}$, let $B_k(G, y)$ denote the induced $\mathcal{Y}$-marked rooted subgraph consisting of vertices within the ball of radius k centered at the root. We say that two $\mathcal{Y}$-marked rooted graphs (G, y) and (G', y') are *isomorphic* if there exists an isomorphism φ from G to G' such that $(y_v)_{v \in V} = (y'_{\varphi(v)})_{v \in V}$. We write $(G, y) \cong (G', y')$ to indicate isomorphism.

Let $\mathcal{G}_*[\mathcal{Y}]$ denote the set of isomorphism classes of $\mathcal{Y}$-marked rooted graphs. We say that a sequence $\{(G_n, y^n)\} \subset \mathcal{G}_*[\mathcal{Y}]$ *converges locally* to $(G, y) \in \mathcal{G}_*[\mathcal{Y}]$ if, for every $k \in \mathbb{N}$ and $\epsilon > 0$, there exists $n_k \in \mathbb{N}$ such that for all $n \geq n_k$ there exists an isomorphism $\varphi : B_k(G_n) \mapsto B_k(G)$ with $\max_{v \in B_k(G_n)} d(y^n_v, y_{\varphi(v)}) < \epsilon$. The space $\mathcal{G}_*[\mathcal{Y}]$ can be equipped with a metric compatible with this notion of convergence, and if $(\mathcal{Y}, d)$ is complete and separable then so is $\mathcal{G}_*[\mathcal{Y}]$ (cf. [4, Lemma 3.4]). An equivalent metric which we will use on occasion is

$$d_*((G, y), (G', y')) = \sum_{k=1}^{\infty} 2^{-k} \left(1 \wedge \inf_{\varphi \in I(B_k(G), B_k(G'))} \max_{v \in B_k(G)} d(y_v, y'_{\varphi(v)}) \right). \tag{1.2}$$

1.3. **Examples of locally convergent graph sequences.** Here we catalog some of the most well known examples of locally converging graphs. For a (finite or countable, locally finite, possibly disconnected) graph $G = (V, E)$ and a vertex $v \in V$, we write $\mathsf{C}_v(G)$ for the connected component of v, that is, the set of $u \in V$ for which there exists a path from v to u. By viewing v as the root, $\mathsf{C}_v(G)$ is then an element of $\mathcal{G}_*$. Note that even if two distinct vertices u and v belong to the same connected component of G, the *rooted* graphs $\mathsf{C}_u(G)$ and $\mathsf{C}_v(G)$ can be non-isomorphic and thus induce distinct elements of $\mathcal{G}_*$. When the graph is finite, we may choose a uniformly random vertex U of G, and we write $\mathsf{C}_{\mathrm{Unif}}(G) := \mathsf{C}_U(G)$ for the resulting $\mathcal{G}_*$-valued random variable. That is, we write $\mathsf{C}_{\mathrm{Unif}}(G)$ for the random connected rooted graph obtained by assigning a root uniformly at random and then isolating the connected component containing this root. We define $\mathsf{C}_v(G, y) := (\mathsf{C}_v(G), y_{\mathsf{C}_v(G)})$ and $\mathsf{C}_{\mathrm{Unif}}(G, y)$ similarly for marked graphs.

Example 1.2. Consider the Erdős-Rényi graph $G_n \sim \mathcal{G}(n, p_n)$, with $\lim_{n \to \infty} n p_n = \theta \in (0, \infty)$. Then $\{\mathsf{C}_{\mathrm{Unif}}(G_n)\}$ converges in law in $\mathcal{G}_*$ to the Galton-Watson tree with offspring distribution Poisson(θ), denoted GW(Poisson(θ)). Similarly, suppose $G_n \sim \mathcal{G}_{n, m_n}$, which means G_n is selected uniformly at random from all (labeled) graphs on n vertices with m_n edges. If $\lim_{n \to \infty} 2m_n/n = \theta \in (0, \infty)$, then again $\{\mathsf{C}_{\mathrm{Unif}}(G_n)\}$ converges to GW(Poisson(θ)). See [5, Proposition 2.6] or [4, Theorem 3.12] for proofs of these facts.

Example 1.3. Given a graphic sequence $d(n) = (d_1(n), \ldots, d_n(n))$, with each $d_i(n)$ a positive integer less than n, let $G_n \sim \mathrm{CM}(n, d(n))$ be a uniformly random graph on n vertices with degree sequence $d(n)$. Alternatively, this may be constructed from the configuration model conditioned to have no multi-edges or self-edges (see [9, Chapter 7]). Suppose the sequence of degree distributions $\{\frac{1}{n} \sum_{i=1}^n \delta_{d_i(n)}\}$ converges to some distribution $\rho \in \mathcal{P}(\mathbb{N}_0)$ with a finite nonzero first moment, and assume also that the first moments converge, $\frac{1}{n} \sum_{i=1}^n d_i(n) \to \sum_{k \in \mathbb{N}_0} k \rho(k)$. Then $\{\mathsf{C}_{\mathrm{Unif}}(G_n)\}$ converges in law in $\mathcal{G}_*$ to the *augmented* or *unimodular Galton-Watson tree with degree distribution* ρ, denoted UGW(ρ) and defined as follows: The root has offspring distribution ρ, and each subsequent generation has an independent number of offspring according to the distribution $\widehat{\rho}$, where $\widehat{\rho}$ is defined by

$$\widehat{\rho}(k) = \frac{(k+1)\rho(k+1)}{\sum_{n \in \mathbb{N}} n \rho(n)}, \quad k \in \mathbb{N}_0. \tag{1.3}$$

Note that $\widehat{\rho} = \rho$ when ρ is Poisson. See [5, Proposition 2.5], [4, Theorem 3.15], or [10, Theorem 4.1] for a derivation of this limit.

Example 1.4. Let G_n denote the uniform κ-regular graph on n vertices, for $\kappa \geq 2$. Then the sequence $\{\mathsf{C}_{\text{Unif}}(G_n)\}$ converges in law in $\mathcal{G}_*$ to the infinite κ-regular tree; this is a well known consequence of the results of [3]. Note that the infinite κ-regular tree is nothing but $\text{UGW}(\delta_\kappa)$.

1.4. **Convergence notions in the local weak sense.** Fix throughout this section a sequence of finite (possibly disconnected) random graphs $\{G_n\}$. Let G be a random element of $\mathcal{G}_*$.

Definition 1.5. We say that $\{G_n\}$ *converges in probability in the local weak sense* to G if

$$\lim_{n\to\infty} \frac{1}{|G_n|} \sum_{v \in G_n} f(\mathsf{C}_v(G_n)) = \mathbb{E}[f(G)], \qquad \text{in probability}, \quad \forall f \in C_b(\mathcal{G}_*), \tag{1.4}$$

where we recall that $\mathsf{C}_v(G_n)$ denotes the connected component of vertex v of G_n, rooted at v.

Note that this definition is meaningful even if the sequence of graphs is non-random, in which case of course the phrase "in probability" in (1.4) is redundant.

Remark 1.6. Because $\mathcal{G}_*$ is a Polish space, a standard argument using a countable convergence-determining set in $C_b(\mathcal{G}_*)$ yields the following equivalent definition: $\{G_n\}$ converges in probability in the local weak sense to G if and only if

$$\lim_{n\to\infty} \frac{1}{|G_n|} \sum_{v \in G_n} \delta_{\mathsf{C}_v(G_n)} = \mathcal{L}(G), \qquad \text{in probability in } \mathcal{P}(\mathcal{G}_*).$$

Remark 1.7. Throughout the paper, if we say that a sequence of random graphs $\{G_n\}$ converges in probability in the local weak sense, it should be understood that we implicitly require that the vertex set of each graph G_n is finite.

The definition of convergence in probability is borrowed from [10, Definition 2.7], where one also defines *converges in distribution* or *in law* in the local weak sense as follows:

Definition 1.8. We say that $\{G_n\}$ *converges in distribution or law in the local weak sense* to G if

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{1}{|G_n|} \sum_{v \in G_n} f(\mathsf{C}_v(G_n))\right] = \mathbb{E}[f(G)], \qquad \forall f \in C_b(\mathcal{G}_*), \tag{1.5}$$

where, recalling that $\mathsf{C}_{\text{Unif}}(G_n)$ denotes the connected component of a uniformly randomly chosen root in G_n, we may write the expectation on the left-hand side of (1.5) as $\mathbb{E}[f(\mathsf{C}_{\text{Unif}}(G_n))]$.

Hence, convergence of $\{G_n\}$ to G in distribution in the local weak sense is equivalent to convergence in law of $\{\mathsf{C}_{\text{Unif}}(G_n)\}$ to G in $\mathcal{G}_*$, and of course convergence in probability in the local weak sense is a stronger property.

Remark 1.9. For each of the examples in Section 1.3, it is known that there is in fact convergence in probability in the local weak sense; see [10, Theorems 3.11 and 4.1].

The above discussion is equally valid for marked graphs. Let $\mathcal{Y}$ be a Polish space. Let $y^n = (y^n_v)_{v \in G_n}$ be random $\mathcal{Y}$-valued marks on the vertices of G_n, and let $y = (y_v)_{v \in G}$ be random $\mathcal{Y}$-valued marks on G.

Definition 1.10. We say that the sequence $\{(G_n, y^n)\}$ *converges in probability in the local weak sense* to (G, y) if

$$\lim_{n\to\infty} \frac{1}{|G_n|} \sum_{v \in G_n} f(\mathsf{C}_v(G_n, y^n)) = \mathbb{E}[f(G, y)], \qquad \text{in probability}, \quad \forall f \in C_b(\mathcal{G}_*[\mathcal{Y}]), \tag{1.6}$$

Once again, convergence of $\{(G_n, y^n)\}$ to (G, y) in probability in the local weak sense implies $\{\mathsf{C}_{\mathrm{Unif}}(G_n, y^n)\}$ converges in law to (G, y) in $\mathcal{G}_*[\mathcal{Y}]$. Remark 1.7 applies also for marked graphs.

Note that the "root mark map" $\mathcal{G}_*[\mathcal{Y}] \ni (G, \varnothing, y) \mapsto y_\varnothing \in \mathcal{Y}$ is continuous. Thus, applying (1.6) with f of the form $f(G, \varnothing, y) = g(y_\varnothing)$ for $g \in C_b(\mathcal{Y})$, we deduce that convergence in probability in the local weak sense implies convergence in probability of the empirical mark distributions:

Lemma 1.11. *If $\{(G_n, y^n)\}$ converges in probability in the local weak sense to $(G, \varnothing, y)$, then the empirical measure sequence $\{\frac{1}{|G_n|}\sum_{v \in G_n} \delta_{y_v^n}\}$ converges in probability to $\mathcal{L}(y_\varnothing)$ in $\mathcal{P}(\mathcal{Y})$.*

Lastly, we state a useful equivalent characterization of convergence in probability in the local weak sense, valid for marked or unmarked graphs. The proof is given in Appendix 2.1.1.

Lemma 1.12. *Suppose $\{G_n\}$ is a sequence of finite (possibly disconnected) random graphs. Suppose $y^n = (y_v^n)_{v \in G_n}$ are (random) marks with values in a Polish space $\mathcal{Y}$, for each $n \in \mathbb{N}$. Let (G, y) be a random element of $\mathcal{G}_*[\mathcal{Y}]$. Assume $|G_n| \to \infty$ in probability. Let U_1^n and U_2^n denote independent vertices that are uniformly distributed on G_n, given G_n. Then $\{(G_n, y^n)\}$ converges in probability in the local weak sense to (G, x) if and only if*

$$\mathbb{E}[g_1(\mathsf{C}_{U_1^n}(G_n, y^n))g_2(\mathsf{C}_{U_2^n}(G_n, y^n))] \to \mathbb{E}[g_1(G, y)]\mathbb{E}[g_2(G, y)], \quad \forall g_1, g_2 \in C_b(\mathcal{G}_*[\mathcal{Y}]). \tag{1.7}$$

1.5. **Examples where graph convergence implies marked graph convergence.** In Section 1.3 and Remark 1.9 we provided illustrative examples of many interesting examples of graphs $\{G_n\}$ that converge in the local weak sense (both in law and in probability). For many of our results, we will require that the sequence of randomly marked random graphs $\{(G_n, Y^n)\}$ converge locally (either in law or in probability), where the random marks $Y^n = (Y_v^n)_{v \in G_n}$ represent random initial conditions taking values in some Polish space $\mathcal{Y}$. It is thus natural to ask if there are important classes of random initial conditions for which the local weak convergence of $\{G_n\}$ implies the local weak convergence of the corresponding randomly $\mathcal{Y}$-marked graphs. It is shown in Corollary 1.17 that this is true when the random initial conditions $Y = (Y_v)_{v \in G}$ are i.i.d. A more general class of initial conditions for which this holds is the class of Gibbs measures, defined below. Throughout, fix the Polish space $\mathcal{Y}$, a reference measure $\lambda \in \mathcal{P}(\mathcal{Y})$ and a bounded continuous function $\psi : \mathcal{Y}^2 \to [0, \infty)$ that serves as a pairwise interaction potential.

Definition 1.13. For each finite graph $G = (V, E)$, the (ψ, λ)-*Gibbs measure* on G is the probability measure $P_G \in \mathcal{P}(\mathcal{Y}^V)$ defined by

$$P_G(d(y_v)_{v \in V}) = \frac{1}{Z^G} \prod_{(u,v) \in E} \psi(y_v, y_u) \prod_{v \in V} \lambda(dy_v),$$

where $Z^G > 0$ is the normalizing constant.

This definition does not make sense for infinite graphs G since Z^G is infinite in that case. Instead, as is standard practice, we use an alternative characterization of P_G in terms of a certain conditional independence or Markov random field property, which then admits a natural extension to locally finite infinite graphs $G = (V, E)$. Given (ψ, λ) as above and a finite set $A \subset V$, as usual let $\partial A := \{u \in V \setminus A : (u, v) \in E \text{ for some } v \in A\}$ denote the boundary of A, and define a map $\mathcal{Y}^{\partial A} \ni y_{\partial A} \mapsto \gamma_A^G(\cdot \,|\, y_{\partial A}) \in \mathcal{P}(\mathcal{Y}^A)$ by

$$\gamma_A^G(dy_A \,|\, y_{\partial A}) = \frac{1}{Z_A^G(y_{\partial A})} \prod_{(u,v) \in E : u \in A,\, v \in A \cup \partial A} \psi(y_v, y_u) \prod_{w \in A} \lambda(dy_w), \tag{1.8}$$

where $Z_A^G(y_{\partial A}) > 0$ is the normalizing constant. Note that for finite G, any random element $Y^G = (Y_v^G)_{v \in G}$ taking values in $\mathcal{Y}^G$ whose law is the (ψ, λ)-Gibbs measure $P_G \in \mathcal{P}(\mathcal{Y}^V)$ satisfies for every finite $A \subset V$,

$$\gamma_A^G(\cdot \,|\, Y_{\partial A}^G) = \mathcal{L}(Y_A^G \,|\, Y_{\partial A}^G) = \mathcal{L}(Y_A^G \,|\, Y_{V\setminus A}^G) \quad a.s. \tag{1.9}$$

It is clear that $\gamma_A^G = \gamma_A^H$ whenever $A \cup \partial A$ is a common subset of the vertex sets of two graphs G and H that induce the same subgraph on $A \cup \partial A$. The observation (1.9) motivates the following definition.

Definition 1.14. For a general (countable, locally finite) graph $G = (V, E)$, the set $\mathrm{Gibbs}(G) = \mathrm{Gibbs}(G, \psi, \lambda) \subset \mathcal{P}(\mathcal{Y}^V)$ of (ψ, λ)-Gibbs measures on G is the set of laws $\mathcal{L}((Y_v^G)_{v \in V})$, where $(Y_v^G)_{v \in V}$ is a random element of $\mathcal{Y}^V$ such that

$$\mathcal{L}(Y_A^G \,|\, Y_{V\setminus A}^G) = \gamma_A^G(\cdot \,|\, Y_{\partial A}^G) \quad a.s.,$$

for each finite set $A \subset V$, where γ_A^G is as defined in (1.9).

Unlike in the finite case, when the graph is infinite, the Gibbs measure may not be unique. However, since the reference measure $\otimes_{v \in V} \lambda$ is invariant under permutations of the vertex set of the graph and the interaction potential ψ is homogeneous in the sense that it is the same on all edges of the graph, it is easy to see from Definition 1.14 that $|\mathrm{Gibbs}(G_1, \psi, \lambda)| = |\mathrm{Gibbs}(G_2, \psi, \lambda)|$ whenever G_1 is isomorphic to G_2 (see also [7, Chapter 5] for related assertions). Therefore, we can define $\mathcal{U} = \mathcal{U}_{\psi,\lambda}$ by

$$\mathcal{U} := \{G \in \mathcal{G}_* : |\mathrm{Gibbs}(G)| = 1\}. \tag{1.10}$$

In other words, $\mathcal{U}$ consists of (isomorphism classes of) locally finite graphs G for which $\mathrm{Gibbs}(G)$ is a singleton. For $G \in \mathcal{U}$, let P_G denote the unique element of $\mathcal{U}$. Note that every finite connected graph belongs to $\mathcal{U}$, so this is consistent with the notation introduced in Definition 1.13. Note that if $\psi \equiv 1$ then we recover the i.i.d. setting, where $P_G = \lambda^G$ for each G and in particular $\mathcal{U} = \mathcal{G}_*$. For any $G \in \mathcal{U}$, let Y^G denote a random element of $\mathcal{Y}^G$ with law P_G, and write (G, Y^G) for the corresponding random element of $\mathcal{G}_*[\mathcal{Y}]$.

We now state key convergence results for Gibbs measures, whose proofs are given in Appendix 2.2 for completeness.

Proposition 1.15. *Suppose $G_n, G \in \mathcal{G}_*$ with $G_n \to G$ in $\mathcal{G}_*$. If $G \in \mathcal{U}$, then with Y^{G_n}, Y^G being random Gibbs configurations as defined above, $\mathcal{L}(G_n, Y^{G_n}) \to \mathcal{L}(G, Y^G)$ in $\mathcal{P}(\mathcal{G}_*[\mathcal{Y}])$.*

Now, if G is a random element of $\mathcal{U}$ with law M, we may define a random element $(G, \mathcal{Y}^G)$ of $\mathcal{G}_*[\mathcal{Y}]$ in the natural way, by first sampling G and then generating Y^G according to the measure P_G. More precisely, the law of (G, Y^G) is determined by the identity

$$\mathbb{E}[f(G, Y^G)] = \int_{\mathcal{U}} \mathbb{E}[f(H, Y^H)]\, M(dH), \quad f \in C_b(\mathcal{G}_*[\mathcal{Y}]).$$

Proposition 1.15 ensures that the integrand is continuous in H on $\mathcal{U}$, so that this is well defined.

Proposition 1.16. *Suppose G is a random element of $\mathcal{G}_*$, with $G \in \mathcal{U}$ a.s. Suppose G_n are finite (possibly disconnected) random graphs such that G_n converges in probability (resp. in law) in the local weak sense to G. Then, with Y^{G_n}, Y^G being random Gibbs configurations as defined above, (G_n, Y^{G_n}) converges in probability (resp. in law) in the local weak sense to (G, Y^G).*

An immediate consequence of Propositions 1.15 and 1.16 is that analogous convergence results hold when the initial marks are i.i.d. with law $\lambda \in \mathcal{P}(\mathcal{Y})$, conditionally on the graphs $\{G_n\}$, as stated below.

Corollary 1.17. *Suppose G is a random element of $\mathcal{G}_*$, and G_n is a sequence of finite (possibly disconnected) random graphs such that G_n converges in probability (resp. in law) in the local weak sense to G. Let $Y^n = (Y^n_v)_{v\in G_n}$ and $Y = (Y_v)_{v\in G}$ be i.i.d. with law λ, given the graphs. Then $\{(G_n, Y^{G_n})\}$ converges in probability (resp. in law) in the local weak sense to (G, Y^G).*

2. Local convergence of marked graphs - Definitions, Results and Proofs

2.1. **Essential properties of the metric space of local convergence.** This subsection develops the essential properties of the space $\mathcal{G}_*[\mathcal{Y}]$ of isomorphism classes of rooted connected marked graphs, introduced in the previous section. Throughout this section, $(\mathcal{Y}, d)$ is a fixed metric space. Some of these results (albeit with a different choice of metric that induces the same topology) can be found in [4, Section 3.2].

Let $I(G, G')$ denote the set of isomorphisms between two graphs $G, G' \in \mathcal{G}_*$. Recall from Section 1.1 that a sequence $\{(G_n, y^n)\} \subset \mathcal{G}_*[\mathcal{Y}]$ *converges locally* to $(G, y) \in \mathcal{G}_*[\mathcal{Y}]$ if, for every $k \in \mathbb{N}$ and $\epsilon > 0$, there exist $N \in \mathbb{N}$ such that for all $n \ge N$ there exists $\varphi \in I(B_k(G_n), B_k(G))$ with $d(y^n_v, y_{\varphi(v)}) < \epsilon$ for all $v \in B_k(G_n)$, where recall that $B_k(G_n)$ represents the induced subgraph of G_n on vertices of G_n that are no greater than distance k from the root. We may endow $\mathcal{G}_*[\mathcal{Y}]$ with either of the following two metrics:

$$d_*((G,y),(G',y')) = \sum_{k=1}^{\infty} 2^{-k} \left(1 \wedge \inf_{\varphi\in I(B_k(G),B_k(G'))} \max_{v\in B_k(G)} d(y_v, y'_{\varphi(v)})\right),$$

$$d_{*,1}((G,y),(G',y')) = \sum_{k=1}^{\infty} 2^{-k} \left(1 \wedge \inf_{\varphi\in I(B_k(G),B_k(G'))} \frac{1}{|B_k(G)|} \sum_{v\in B_k(G)} d(y_v, y'_{\varphi(v)})\right),$$

where the infimum of the empty set is understood to be infinite. We will show in Lemma 2.2 that these are genuine metrics on $\mathcal{G}_*[\mathcal{Y}]$. The following proposition confirms first that they are indeed compatible with the aforementioned notion of local convergence.

Proposition 2.1. *Let $(G, y), (G_n, y^n) \in \mathcal{G}_*[\mathcal{Y}]$, for $n \in \mathbb{N}$. The following are equivalent:*

(1) (G_n, y^n) converges locally to (G, y).
(2) $d_*((G_n, y^n), (G, y)) \to 0$.
(3) $d_{*,1}((G_n, y^n), (G, y)) \to 0$.

Proof. Clearly $d_{*,1} \le d_*$, so (2) $\Rightarrow$ (3). To prove (1) $\Rightarrow$ (2), suppose (G_n, y^n) converges locally to (G, y). Fix $\epsilon > 0$ and $k \in \mathbb{N}$ such that $2^{1-k} \le \epsilon$. Find n_k such that for all $n \ge n_k$ there exists $\varphi_n \in I(B_k(G_n), B_k(G))$ with $d(y^n_v, y_{\varphi_n(v)}) < 2^{-k}$ for all $v \in B_k(G_n)$. Note that for $j < k$ the restriction $\varphi_n|_{B_j(G_n)}$ belongs to $I(B_j(G_n), B_j(G))$. We deduce that, for $n \ge n_k$,

$$\begin{aligned} d_*((G_n,y^n),(G,x)) &< \sum_{j=1}^{k} 2^{-j}2^{-k} + \sum_{j=k+1}^{\infty} 2^{-j} \left(1 \wedge \inf_{\varphi\in I(B_j(G),B_j(G'))} \max_{v\in B_j(G)} d(y_v, y'_{\varphi(v)})\right) \\ &\le 2^{-k} + 2^{-k} \le \epsilon. \end{aligned}$$

Finally, to prove (3) $\Rightarrow$ (1), fix $k \in \mathbb{N}$ and $\epsilon > 0$. Choose $M \in \mathbb{N}$ such that $2^{-M} < \epsilon/|B_k(G)|$ and $M \ge k$. Find N such that $d_{*,1}((G_n, y^n), (G, y)) < 2^{-2M}$ for all $n \ge N$. Then

$$\inf_{\varphi\in I(B_j(G),B_j(G_n))} \frac{1}{|B_j(G)|} \sum_{v\in B_j(G)} d(y_v, y^n_{\varphi(v)}) < 2^{j-2M} \le 2^{-M}, \qquad n \ge N,\ j \le M.$$

In particular, choosing $j = k$, we may thus find for each n some $\varphi_n \in I(B_k(G), B_k(G_n))$ such that

$$\frac{1}{|B_k(G)|} \sum_{v \in B_k(G)} d(y_v, y^n_{\varphi(v)}) < 2^{-M}.$$

Bounding the maximum by the sum,

$$\max_{v \in B_k(G)} d(y_v, y^n_{\varphi(v)}) < 2^{-M}|B_k(G)| \leq \epsilon.$$

In summary, we have shown that for each $k \in \mathbb{N}$ and $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ there exists $\varphi_n \in I(B_k(G), B_k(G_n))$ such that $\max_{v \in B_k(G)} d(y_v, y^n_{\varphi(v)}) < \epsilon$. This shows that $(G_n, y^n) \to (G, y)$ locally, and the proof is complete. □

Lemma 2.2. *$(\mathcal{G}_*[\mathcal{Y}], d_*)$ and $(\mathcal{G}_*[\mathcal{Y}], d_{*,1})$ are metric spaces.*

Proof. We first check that d_* is a metric. Symmetry is clear, as is the fact that $(G, y) \cong (G', y')$ implies $d_*((G, y), (G', y')) = 0$. Conversely, if $d_*((G, y), (G', y')) = 0$, we show that (G, y) and (G', y') are isomorphic as follows: Find a sequence $\varphi_k \in I(B_k(G), B_k(G'))$ such that $y_v = y'_{\varphi_k(v)}$ for all $v \in B_k(G)$. Extend each φ_k arbitrarily to a function from G to G', and view each φ_k as an element of the space $(V')^V$. Endowing V' and V with the discrete topology, we may equip $(V')^V$ with the topology of pointwise convergence. The sequence (φ_n) is pre-compact in this topology since $\varphi_n|_{B_k(G)} \subset B_k(G')^V$ for each $n \geq k$, so we may find a subsequential limit point $\varphi : V \to V'$. The restriction $\varphi|_{B_k(G)}$ belongs to $I(B_k(G), B_k(G'))$ for each k, and it follows that φ must be an isomorphism from G to G'. Moreover, we must have $y_v = y'_{\varphi(v)}$ for all $v \in B_k(G)$, for all k, and we conclude that φ is an isomorphism from (G, y) to (G', y').

Next, note that $d_{*,1}((G, y), (G', y')) = 0$ if and only if $d_*((G, y), (G', y')) = 0$. Therefore $d_{*,1}$ is also a metric. □

The following lemma is taken from [4, Lemma 3.4].

Lemma 2.3. *If $\mathcal{Y}$ is a Polish space, then so is $\mathcal{G}_*[\mathcal{Y}]$.*

2.1.1. *Auxiliary results.* With the essential properties of the metric space $(\mathcal{G}_*[\mathcal{Y}], d_*)$ now established, we now establish two auxiliary results. The first addresses the question of convergence of empirical measures.

Proposition 2.4. *Suppose $(\mathcal{Y}, d)$ is a complete, separable metric space. Let $(G, y), (G_n, y^n) \in \mathcal{G}_*[\mathcal{Y}]$, and assume G and G_n are finite graphs. Define the empirical measures*

$$\mu^G = \frac{1}{|G|} \sum_{v \in G} \delta_{y_v}, \qquad \mu_n = \frac{1}{|G_n|} \sum_{v \in G_n} \delta_{y^n_v}.$$

If $(G_n, y^n) \to (G, y)$ in $\mathcal{G}_*[\mathcal{Y}]$, then $\mu_n \to \mu^G$ in $\mathcal{P}(\mathcal{Y})$.

Proof. Fix finite graphs G, G_n in $\mathcal{G}_*$. Consider the 1-Wasserstein (Kantorovich) metric,

$$W_1(m, m') = \sup\left\{ \int_{\mathcal{X}} f\, d(m - m') : f : \mathcal{X} \to \mathbb{R},\ |f(x) - f(y)| \leq d(x, y)\, \forall x, y \in \mathcal{X} \right\}.$$

It is well known that convergence in this metric implies weak convergence. For any (rooted connected) graph $G' = (V', E', \varnothing') \in \mathcal{G}_*$ let $R(G') = \inf\{n \geq 0 : G' = B_n(G')\}$, and note that $R(G')$ is simply the distance from the root to the furthest vertex. A graph $G' \in \mathcal{G}_*$ is finite if and only if $R(G') < \infty$. Moreover, $G' = B_{R(G')}(G') = B_r(G')$ for any $r \geq R(G')$. Because the graph is connected, it is also clear that if $B_r(G') = B_s(G')$ for some $s > r$, then there are no vertices that are at a distance greater than r from the root, and so $G' = B_r(G')$ and $R(G') \leq r$.

Now, let $r = 2R(G)$. Let $\epsilon > 0$. The assumed convergence $(G_n, y^n) \to (G, y)$ implies the existence of $N \in \mathbb{N}$ such that for all $n \geq N$ there exists $\varphi_n \in I(B_r(G), B_r(G_n))$ such that $\max_{v \in B_r(G)} d(y_v, y^n_{\varphi_n(v)}) < \epsilon$. Now, since $G = B_r(G) = B_{R(G)}(G)$, by isomorphism we must have $B_r(G_n) = B_{R(G)}(G_n)$. From the argument of the previous paragraph we deduce that $G_n = B_r(G_n)$ and $R(G_n) = R(G)$. Thus φ_n is an isomorphism from G to G_n, and

$$W_1(\mu_n, \mu) = \sup_f \frac{1}{|G|} \sum_{v \in G} \Big(f(y_v) - f(y^n_{\varphi_n(v)}) \Big) \leq \frac{1}{|G|} \sum_{v \in G} d(y_v, y^n_{\varphi_n(v)}) < \epsilon. \qquad \square$$

We now present the proof of Lemma 1.12 stated in Section 1.4, which provides equivalent characterizations of convergence in probability in the local weak sense.

Proof of Lemma 1.12. The proof is similar to that of the Sznitman-Tanaka theorem [8, Proposition 2.2(i)]. A simple and well known argument shows that the total variation distance between $\mathcal{L}((U^n_1, U^n_2) \,|\, G_n)$ and $\mathcal{L}((\pi_n(1), \pi_n(2)) \,|\, G_n)$ is no more than $2/|G_n|$ on the set $|G_n| \geq 2$, where π_n is a uniformly random permutation of the vertex set of G_n (given G_n, and assuming without loss of generality that the vertex set of G_n is $\{1, \ldots, |G_n|\}$). Since $|G_n| \to \infty$ in probability, we deduce that the total variation distance between $\mathcal{L}(U^n_1, U^n_2)$ and $\mathcal{L}(\pi_n(1), \pi_n(2))$ vanishes. Thus, (1.7) is equivalent to

$$\mathbb{E}[g_1(\mathsf{C}_{\pi_n(1)}(G_n, y^n)) g_2(\mathsf{C}_{\pi_n(2)}(G_n, y^n))] \to \mathbb{E}[g_1(G, y)]\mathbb{E}[g_2(G, y)], \quad \forall g_1, g_2 \in C_b(\mathcal{G}_*[\mathcal{Y}]). \tag{2.1}$$

Let $\mu_n := \frac{1}{|G_n|} \sum_{v \in G_n} \delta_{\mathsf{C}_{\pi_n(v)}(G_n, y^n)}$. Since the (conditional) joint law $\mathcal{L}\big((\mathsf{C}_{\pi_n(v)}(G_n, y^n))_{v \in G_n} | G_n\big)$ is exchangeable, for $g_1, g_2 \in C_b(\mathcal{G}_*[\mathcal{Y}])$ we have

$$\begin{aligned}
\mathbb{E}\left[\langle \mu_n, g_1 \rangle \langle \mu_n, g_2 \rangle\right] &= \mathbb{E}\left[\frac{1}{|G_n|^2} \sum_{u, v \in G_n} g_1(\mathsf{C}_{\pi_n(u)}(G_n, y^n)) g_2(\mathsf{C}_{\pi_n(v)}(G_n, y^n)) \right] \\
&= \mathbb{E}\Bigg[\frac{|G_n| - 1}{|G_n|} g_1(\mathsf{C}_{\pi_n(1)}(G_n, y^n)) g_2(\mathsf{C}_{\pi_n(2)}(G_n, y^n)) \\
&\qquad + \frac{1}{|G_n|} g_1(\mathsf{C}_{\pi_n(1)}(G_n, y^n)) g_2(\mathsf{C}_{\pi_n(1)}(G_n, y^n)) \Bigg].
\end{aligned} \tag{2.2}$$

Now suppose that (1.7), or equivalently (2.1), holds. Let $f \in C_b(\mathcal{G}_*[\mathcal{Y}])$, and take $g_i(\cdot) := f(\cdot) - \mathbb{E}[f(G, y)]$ for $wi = 1, 2$. Then the right-hand side of (2.2) converges to $\mathbb{E}[g_1(G, y)]\mathbb{E}[g_2(G, y)] = 0$, and we deduce that

$$\mathbb{E}\left[(\langle \mu_n, f \rangle - \mathbb{E}[f(G, y)])^2 \right] = \mathbb{E}[\langle \mu_n, g_1 \rangle \langle \mu_n, g_2 \rangle] \to 0.$$

As this holds for arbitrary f, we deduce that

$$\lim_{n \to \infty} \frac{1}{|G_n|} \sum_{v \in G_n} \delta_{\mathsf{C}_v(G_n, y^n)} = \lim_{n \to \infty} \mu_n = \mathcal{L}(G, y), \qquad \text{in } \mathcal{P}(\mathcal{G}_*[\mathcal{Y}]), \text{ in probability.} \tag{2.3}$$

Note that the first identity is just the definition of μ_n, upon removing the permutation. Thus, (2.3) is precisely the convergence in probability in the local weak sense of (G_n, y) to (G, y), which completes the proof of the "if" part of the claim.

To prove the converse, we assume (2.3) holds and deduce (2.1) as follows. Note that (2.3) implies $\mathbb{E}\left[\langle \mu_n, g_1 \rangle \langle \mu_n, g_2 \rangle\right]$ converges to $\mathbb{E}[g_1(G, y)]\mathbb{E}[g_2(G, y)]$, whereas the right-hand side of (2.2) clearly has the same $n \to \infty$ limit as the left-hand side of (2.1) since $|G_n| \to \infty$. $\square$

2.2. **Proofs of local weak convergence of Gibbs measures.** The goal of this section is to prove the results in Section 1.5. A number of prior works, such as [5, 6], have studied Gibbs measures on locally converging (sparse) graph sequences and Lemma 2.5 on the convergence of the whole particle configuration is well known in a more general context (see [7]), but we include it here for completeness.

Although we have focused our attention on *factor models* with pairwise interactions, the same arguments extend easily to bounded-range interactions. Recall the definitions given in Section 1.5, and fix the pair (ψ, λ) as defined therein. Also, recall that given a Polish space $\mathcal{Y}$ and a graph $G = (V, E)$, $P \in \mathcal{P}(\mathcal{Y}^V)$ is said to be a *Markov random field* with respect to G if for every finite $A \subset V$,

$$P(y_A \,|\, y_{V\setminus A}) = P(y_A \,|\, y_{\partial A}) \quad \text{for } P\text{-a.e. } y_{V\setminus A} \in \mathcal{Y}^{V\setminus A}. \tag{2.4}$$

The first step toward the proofs of Propositions 1.15 and 1.16 is the following lemma, which is inspired by [7, Proposition 7.11].

Lemma 2.5. *Suppose $G = (V, E) \in \mathcal{U}$. Let P_G be the unique (ψ, λ)-Gibbs measure, and let A_n be any increasing sequence of finite sets with $\cup_n A_n = V$. Then, for any $m \in \mathbb{N}$ and $f \in C_b(\mathcal{Y}^{A_m})$, we have*

$$\lim_{n\to\infty} \sup_{y_{\partial A_n} \in \mathcal{Y}^{\partial A_n}} \left| \int_{\mathcal{Y}^{A_n}} f(y_{A_m})\, \gamma^G_{A_n}(dy_{A_n} \,|\, y_{\partial A_n}) - \int_{\mathcal{Y}^V} f(\bar{y}_{A_m})\, P_G(d\bar{y}_V) \right| = 0.$$

Recall that the definition of the kernel $\gamma^G_A(dy_A \,|\, y_{\partial A})$ is given pointwise in (1.8), in terms of the continuous interaction function ψ. Because we work with this particular version of the conditional probability measures, it makes sense that Lemma 2.5 is stated in terms of a supremum rather than an essential supremum.

Proof of Lemma 2.5. We first note that $y_{\partial A} \mapsto \gamma^G_A(\cdot \,|\, y_{\partial A})$ is continuous with respect to weak convergence, for any finite graph G and nonempty finite set of vertices A, because ψ is bounded and continuous. Moreover, because ψ is bounded, we have

$$\sup_{y \in \mathcal{Y}^V} \frac{d\gamma^G_A(\cdot \,|\, y_{\partial A})}{d\lambda^A}(y_A) < \infty,$$

In particular, this readily implies that

$$\{\gamma^G_A(\cdot \,|\, y_{\partial A}) : y_{\partial A} \in \mathcal{Y}^{\partial A}\} \subset \mathcal{P}(\mathcal{Y}^A) \quad \text{is tight for each } G \text{ and } A. \tag{2.5}$$

Now suppose that, in contradiction to the assertion of the lemma, there exist an increasing sequence of finite sets A_n with $\cup_n A_n = V$, $m \in \mathbb{N}$, $f \in C_b(\mathcal{Y}^{A_m})$, $\epsilon > 0$, and $y^n_{\partial A_n} \in \mathcal{Y}^{\partial A_n}$ such that

$$\left| \int_{\mathcal{Y}^{A_n}} f(y_{A_m})\, \gamma^G_{A_n}(dy_{A_n} \,|\, y^n_{\partial A_n}) - \int_{\mathcal{Y}^V} f(y_{A_m})\, P_G(dy_V) \right| \geq \epsilon, \quad \forall n \geq m. \tag{2.6}$$

Define $P^n \in \mathcal{P}(\mathcal{Y}^V)$ by setting

$$P^n(dy_V) = \gamma^G_{A_n}(dy_{A_n} \,|\, y^n_{\partial A_n}) \prod_{v \in V\setminus A_n} \lambda(dy_v).$$

Then it is easy to verify that P^n is a Markov random field with respect to G, in the sense that (2.4) holds when P is replaced with P^n. Moreover, as a consequence of (2.5), the sequence (P^n) is tight and thus has a weak limit point, say $P \in \mathcal{P}(\mathcal{Y}^V)$. By (2.6), we have

$$\left| \int_{\mathcal{Y}^V} f(y_{A_m})\, (P - P_G)(dy_V) \right| \geq \epsilon. \tag{2.7}$$

Now, let (P^{n_k}) denote a subsequence of (P^n) that converges weakly to P. Also, let $Y^k = (Y^k_v)_{v\in V}$ and $Y = (Y_v)_{v\in V}$ be random $\mathcal{Y}^V$-valued elements with laws P^{n_k} and P, respectively. Consider disjoint finite sets $B, C \subset V$ and $g \in C_b(\mathcal{Y}^B)$, $h \in C_b(\mathcal{Y}^C)$. Then, first using the weak convergence of (P^{n_k}) to P, then the Markov random field property of P^n, the definition of P^{n_k}, and finally the continuity of γ^G_B, we have

$$\begin{aligned}\mathbb{E}[g(Y_B)h(Y_C)] &= \lim_{k\to\infty} \mathbb{E}[g(Y^k_B)h(Y^k_C)]\\ &= \lim_{k\to\infty} \mathbb{E}[\mathbb{E}[g(Y^k_B)\,|\,Y^k_{V\setminus B}]h(Y^k_C)]\\ &= \lim_{k\to\infty} \mathbb{E}[\langle\gamma^G_B(\cdot\,|\,Y^k_{\partial B}),\, g\rangle h(Y^k_C)]\\ &= \mathbb{E}[\langle\gamma^G_B(\cdot\,|\,Y_{\partial B}),\, g\rangle h(Y_C)].\end{aligned}$$

Because B and C are arbitrary finite subsets of V, this is enough to conclude that P belongs to $\mathrm{Gibbs}(G) = \mathrm{Gibbs}(G, \psi, \lambda)$. Since $G \in \mathcal{U}$, this implies $P = P_G$, which contradicts (2.7). □

The following Lemma includes Proposition 1.15 as a special case (by taking G^2_n to be an independent copy of G^1_n and G_n to be the disjoint union of G^1_n and G^2_n), and it will also be useful in proving Proposition 1.16:

Lemma 2.6. *For $n \in \mathbb{N}$, let G_n be a finite (possibly disconnected) random graph, and for $i = 1, 2$, let o^i_n be a (random) vertex in G_n, and let G^i_n be an induced (random) subgraph of G_n rooted at o^i_n. Assume $\mathcal{L}(G^1_n, G^2_n) \to \mathcal{L}(G^1, G^2)$ in $\mathcal{P}(\mathcal{G}_* \times \mathcal{G}_*)$ for some random elements G^1, G^2 of $\mathcal{U}$, and assume also that $d_{G_n}(o^1_n, o^2_n) \to \infty$ as $n \to \infty$ in probability. Then, for random elements Y^{G_n}, Y^{G^1}, and Y^{G^2} with laws P_{G_n}, P_{G^1}, and P_{G^2}, respectively, we have*

$$\mathcal{L}\big((G^1_n, Y^{G_n}_{G^1_n}), (G^2_n, Y^{G_n}_{G^2_n})\big) \to \mathcal{L}(G^1, Y^{G^1}) \times \mathcal{L}(G^2, Y^{G^2}), \quad \text{in } \mathcal{G}_*[\mathcal{Y}] \times \mathcal{G}_*[\mathcal{Y}].$$

Proof. By the Skorohod representation theorem, we may assume that G_n, (G^1_n, o^1_n), and (G^2_n, o^2_n) are non-random. Fix $r \in \mathbb{N}$ and $f_1, f_2 \in C_b(\mathcal{G}_*[\mathcal{Y}])$ with $|f_1|, |f_2| \le 1$. Recall that $B_r(G)$ denotes the ball of radius r around the root in G, and we similarly write $B_r(G, y)$ for a marked graph. It suffices to show that

$$\lim_{n\to\infty} \mathbb{E}\left[f_1(B_r(G^1_n, Y^{G_n}_{G^1_n}))f_2(B_r(G^2_n, Y^{G_n}_{G^2_n}))\right] = \mathbb{E}\left[f_1(B_r(G^1, Y^{G^1}))\right]\mathbb{E}\left[f_2(B_r(G^2, Y^{G^2}))\right]. \quad (2.8)$$

Let $\epsilon > 0$. We may define a function $\widehat{f}_i \in C_b(\mathcal{Y}^{B_r(G^i)})$ by $\widehat{f}_i(y) := f_i(B_r(G^i, y))$, for $i = 1, 2$. By Lemma 2.5 we may find $\ell > r$ such that, for each $i = 1, 2$,

$$\sup_{y\in\mathcal{Y}^{\partial B_\ell(G^i)}} \left|\int_{\mathcal{Y}^{B_\ell(G^i)}} \widehat{f}_i(y_{B_r(G^i)})\,\gamma^{G^i}_{B_\ell(G^i)}(dy_{B_\ell(G^i)}\,|\,y_{\partial B_\ell(G^i)}) - \int_{\mathcal{Y}^{G^i}} \widehat{f}(y_{B_r(G^i)})\,P_{G^i}(dy)\right| \le \epsilon. \quad (2.9)$$

For each $i = 1, 2$, since $G^i_n \to G$, we may find $N < \infty$ such that for all $n \ge N$ there exists an isomorphism $\varphi^i_n : B_{\ell+1}(G^i) \to B_{\ell+1}(G^i_n)$. For any positive integer $m \le \ell + 1$ we may also view φ^i_n as a dual map $\mathcal{Y}^{B_m(G^i_n)} \to \mathcal{Y}^{B_m(G^i)}$ by setting

$$\varphi^i_n y = (y_{\varphi^i_n(v)})_{v\in B_m(G^i)}, \qquad \text{for } y = (y_v)_{v\in B_m(G^i_n)}.$$

For $\bar{y} \in \mathcal{Y}^{\partial B_\ell(G^i_n)}$, we have

$$\begin{aligned}\mathbb{E}\left[\widehat{f}_i(\varphi^i_n Y^{G_n}_{B_r(G^i_n)})\,|\,Y^{G_n}_{\partial B_\ell(G^i_n)} = \bar{y}\right] &= \int_{\mathcal{Y}^{B_\ell(G^i_n)}} \widehat{f}_i(\varphi^i_n y_{B_r(G^i_n)})\,\gamma^{G^i_n}_{B_\ell(G^i_n)}(dy_{B_\ell(G^i_n)}\,|\,\bar{y})\\ &= \int_{\mathcal{Y}^{B_\ell(G^i)}} \widehat{f}_i(y_{B_r(G^i)})\,\gamma^{G^i}_{B_\ell(G^i)}(dy_{B_\ell(G^i)}\,|\,\varphi^i_n\bar{y}),\end{aligned}$$

and thus (2.9) implies

$$\sup_{\bar{y}\in\mathcal{Y}^{\partial B_\ell(G_n^i)}} \left|\mathbb{E}[\widehat{f}_i(\varphi_n^i Y^{G_n}_{B_r(G_n^i)}) \,|\, Y^{G_n}_{\partial B_\ell(G_n^i)} = \bar{y}] - \mathbb{E}[\widehat{f}_i(Y^{G^i}_{B_r(G^i)})]\right| \le \epsilon. \tag{2.10}$$

By assumption we may choose n large enough so that $d_{G_n}(o_n^1, o_n^2) \ge 2\ell$. Then, use the fact that Y^{G_n} is a Markov random field over the graph G_n and the fact that $B_r(G_n^1)$ and $B_r(G_n^2)$ are disjoint, to get

$$\begin{aligned}&\mathbb{E}[\widehat{f}_1(\varphi_n^1 Y^{G_n}_{B_r(G_n^1)})\widehat{f}_2(\varphi_n^2 Y^{G_n}_{B_r(G_n^2)})] \\ &\quad = \mathbb{E}\Big[\mathbb{E}[\widehat{f}_1(\varphi_n^1 Y^{G_n}_{B_r(G_n^1)}) \,|\, Y^{G_n}_{\partial B_\ell(G_n^1)}]\,\mathbb{E}[\widehat{f}_2(\varphi_n^2 Y^{G_n}_{B_r(G_n^2)}) \,|\, Y^{G_n}_{\partial B_\ell(G_n^2)}]\Big].\end{aligned}$$

Combine this with (2.10), and recall that $|f_i| \le 1$, to obtain

$$\left|\mathbb{E}[\widehat{f}_1(\varphi_n^1 Y^{G_n}_{B_r(G_n^1)})\widehat{f}_2(\varphi_n^2 Y^{G_n}_{B_r(G_n^2)})] - \mathbb{E}[\widehat{f}_1(Y^{G^1}_{B_r(G^1)})]\mathbb{E}[\widehat{f}_2(Y^{G^2}_{B_r(G^2)})]\right| \le 2\epsilon,$$

for sufficiently large n. Plugging in the definitions of $\widehat{f}_i$ and φ_n^i, this becomes

$$\left|\mathbb{E}[f_1(B_r(G_n^1, Y^{G_n}_{G_n^1}))f_2(B_r(G_n^2, Y^{G_n}_{G_n^2}))] - \mathbb{E}[f_2(B_r(G^1, Y^{G^1}))]\mathbb{E}[f_2(B_r(G^2, Y^{G^2}))]\right| \le 2\epsilon,$$

for n large. Since ϵ was arbitrary, this implies (2.8). □

Proof of Proposition 1.16. We first prove the "in law" case. Note that the convergence of $G_n \to G$ (resp. $(G_n, Y^{G_n}) \to (G, Y^G)$) in distribution in the local weak sense is equivalent to the convergence in law of $\mathsf{C}_{U^n}(G_n) \to G$ in $\mathcal{G}_*$ (resp. $\mathsf{C}_{U^n}(G_n, Y^{G_n}) \to (G, Y^G)$ in $\mathcal{G}_*[\mathcal{Y}]$), where U^n is a uniform random vertex in G_n. The "in law" case then follows immediately from Proposition 1.15 via continuous mapping or marginalization.

Next we prove the "in probability" case. By Lemma 1.12, we know that

$$\mathcal{L}(\mathsf{C}_{U_1^n}(G_n), \mathsf{C}_{U_2^n}(G_n)) \to \mathcal{L}(G) \times \mathcal{L}(G), \quad \text{in } \mathcal{P}(\mathcal{G}_* \times \mathcal{G}_*),$$

where U_1^n, U_2^n are independent uniform random vertices in G_n. Because $G_n \to G$ in probability in the local weak sense, it is known from [10, Corollary 2.13] that $d_{G_n}(U_1^n, U_2^n) \to \infty$. By passing to a Skorohod representation, we may assume the limits are all almost sure, and then invoke Lemma 2.6 to deduce that

$$\mathcal{L}(\mathsf{C}_{U_1^n}(G_n, Y^{G_n}), \mathsf{C}_{U_2^n}(G_n, Y^{G_n})) \to \mathcal{L}(G, Y^G) \times \mathcal{L}(G, Y^G), \quad \text{in } \mathcal{P}(\mathcal{G}_*[\mathcal{Y}] \times \mathcal{G}_*[\mathcal{Y}]).$$

By Lemma 1.12, this is equivalent to the claim. □

Corollary 2.7. *Suppose G is a random element of $\mathcal{U}$. Suppose $\{G_n\}$ is a sequence of finite (possibly disconnected) random graphs. Let $H_n \subset G_n$ be random induced subgraphs, and let $A \subset \mathcal{G}_*$ be a Borel set with $\mathbb{P}(G \in A) > 0$. Suppose H_n converges in probability in the local weak sense to a random element $\widetilde{H}$ of $\mathcal{G}_*$ with $\mathcal{L}(\widetilde{H}) = \mathcal{L}(G \,|\, G \in A)$. Then, given random elements Y^{G_n} and Y^G with laws P_{G_n} and P_G, respectively, the sequence of marked random graphs $(H_n, Y^{G_n}_{H_n})$ converges in probability in the local weak sense to the marked random graph with law $\mathcal{L}((G, Y^G) \,|\, G \in A)$.*

Proof. By Proposition 1.16, $(H_n, Y^{G_n}_{H_n})$ converges in probability in the local weak sense to $(\widetilde{H}, Y^{\widetilde{H}})$. So we must only argue that $\mathcal{L}(\widetilde{H}, Y^{\widetilde{H}}) = \mathcal{L}((G, Y^G) \,|\, G \in A)$. But this is easy to see: recalling that $\mathcal{U}$ is the collection of graphs on which the Gibbs measure is unique, there exists a map $\Phi : \mathcal{U} \to \mathcal{G}_*[\mathcal{Y}]$, which is in fact continuous by Proposition 1.15, such that

$\mathcal{L}(H, Y^H) = \mathcal{L}(\Phi(H))$ for each $H \in \mathcal{U}$. Together with the assumption $\mathcal{L}(\widetilde{H}) = \mathcal{L}(G \,|\, G \in A)$, this implies the desired result:

$$\mathcal{L}(\widetilde{H}, Y^{\widetilde{H}}) = \mathcal{L}(\Phi(\widetilde{H})) = \mathcal{L}(\Phi(G) \,|\, G \in A) = \mathcal{L}((G, Y^G) \,|\, G \in A).$$

□

References

[1] D. Aldous and J.M. Steele, *The objective method: probabilistic combinatorial optimization and local weak convergence*, Probability on discrete structures, 2004, pp. 1–72.

[2] I. Benjamini and O. Schramm, *Recurrence of distributional limits of finite planar graphs*, Electronic Journal of Probability **6** (2001).

[3] B. Bollobás, *A probabilistic proof of an asymptotic formula for the number of labelled regular graphs*, European Journal of Combinatorics **1** (1980), no. 4, 311–316.

[4] C. Bordenave, *Lecture notes on random graphs and probabilistic combinatorial optimization*, 2016.

[5] A. Dembo and A. Montanari, *Gibbs measures and phase transitions on sparse random graphs*, Brazilian Journal of Probability and Statistics **24** (2010), no. 2, 137–211.

[6] A. Dembo, A. Montanari, and N. Sun, *Factor models on locally tree-like graphs*, The Annals of Probability **41** (2013), no. 6, 4162–4213.

[7] H.-O. Georgii, *Gibbs measures and phase transitions*, Vol. 9, Walter de Gruyter, 2011.

[8] A.-S. Sznitman, *Topics in propagation of chaos*, Ecole d'Eté de Probabilités de Saint-Flour XIX—1989 (1991), 165–251.

[9] R. van der Hofstad, *Random graphs and complex networks*, Vol. 1, Cambridge University Press. Available at https://www.win.tue.nl/ rhofstad/NotesRGCN.pdf, 2016.

[10] ———, *Random graphs and complex networks, volume 2*, Available at https://www.win.tue.nl/ rhofstad/NotesRGCNII_11_07_2020.pdf, 2020.