Free boundary problems and branching particle systems

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Branching-selection systems:

- Particle systems: particles branch and move in space.
  Killing rule keeps number of particles constant.

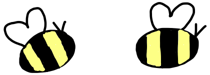
- Toy models for a population under selection.
  Location of a particle (= individual) represents its evolutionary fitness.

- Introduced by Brunet and Derrida in 1990s.
  Recent results and lots of open conjectures about long-term behaviour.

Overview:

1. N-particle branching Brownian motion (N-BBM)
   and related free boundary problem

2. Brownian bees - long-term behaviour



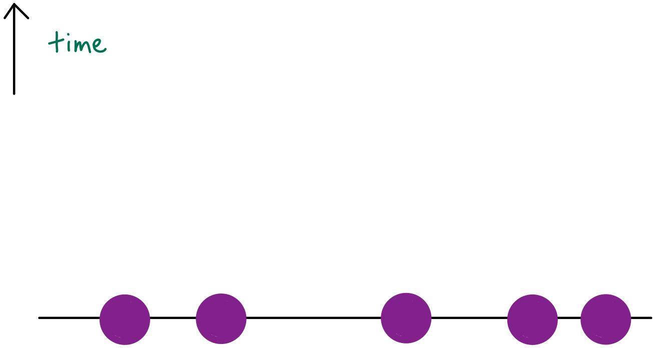
3. N-particle branching random walk
   Results and conjectures about long-term behaviour.

Focus on probabilistic ideas in proofs + how use PDE results.

N-particle branching Brownian motion (N-BBM)

- N particles move in $\mathbb{R}$ according to independent Brownian motions.
- Each particle, independently, branches into two particles after an Exp(1) time.
- Each time a particle branches, the leftmost particle in the system is killed.

N particles in the system at all times.



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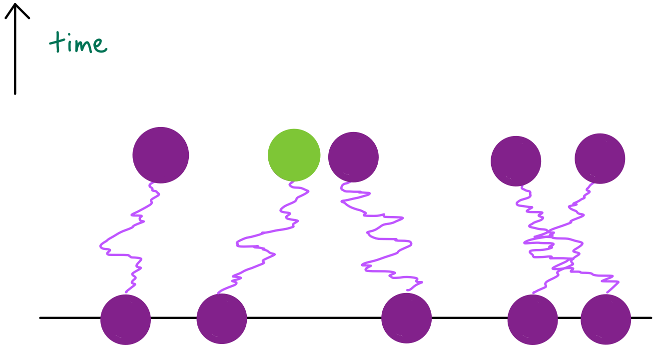

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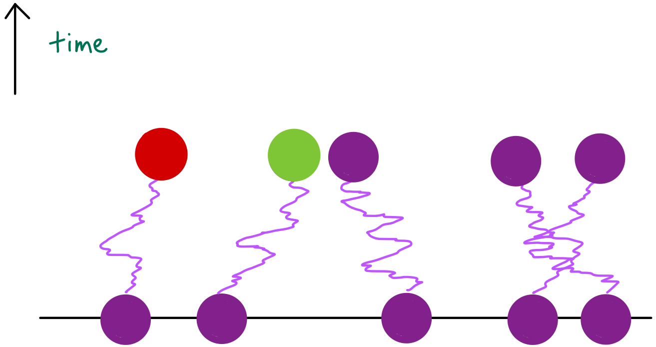

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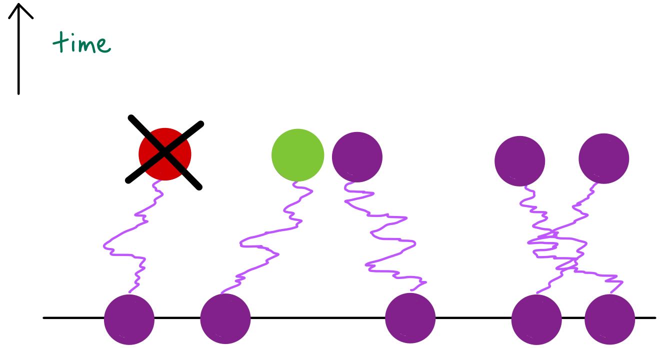

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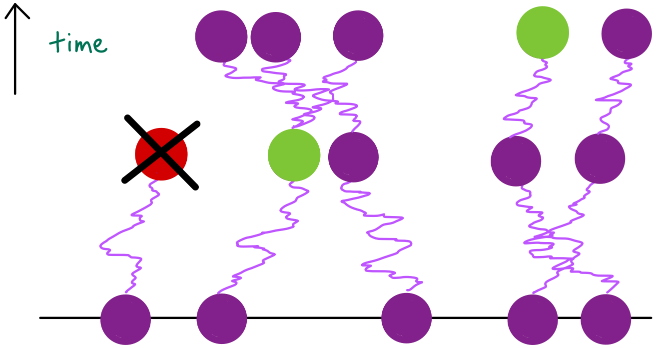

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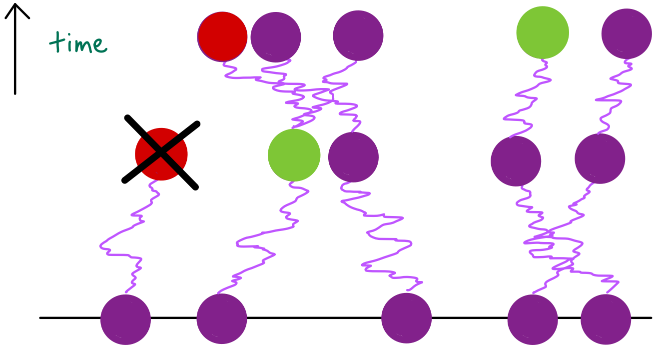

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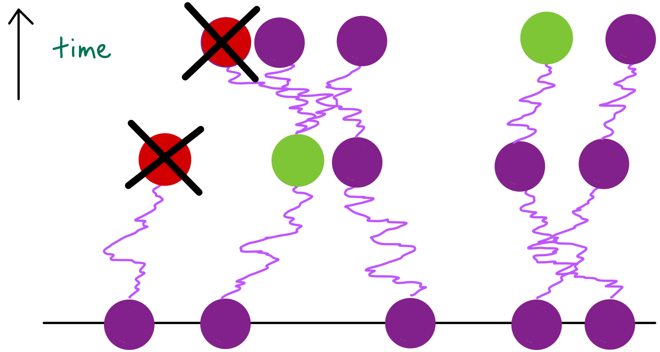

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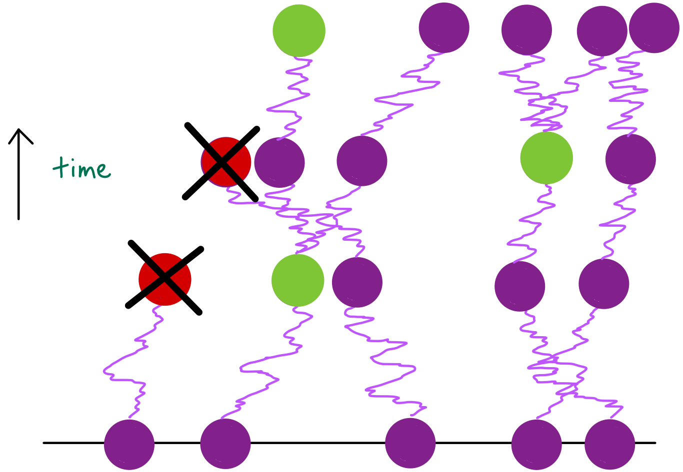

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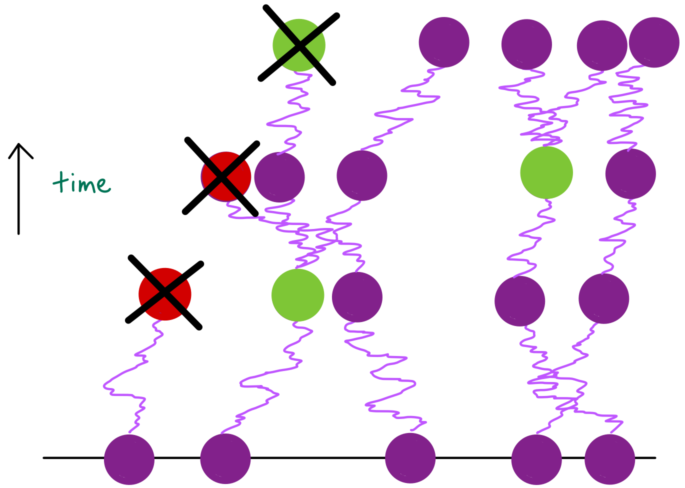

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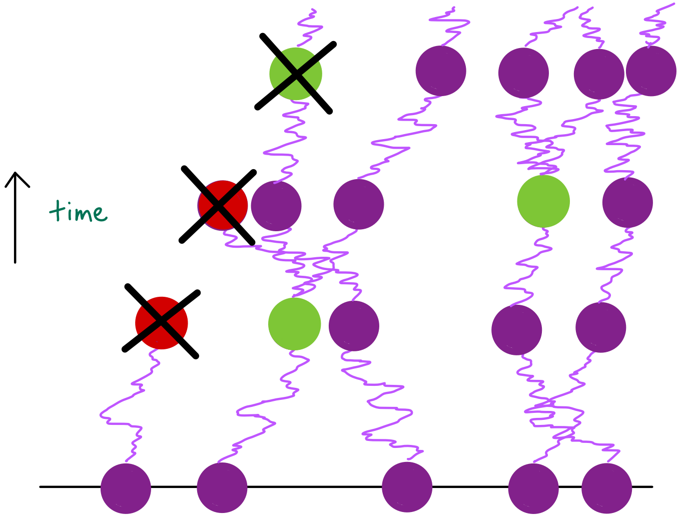

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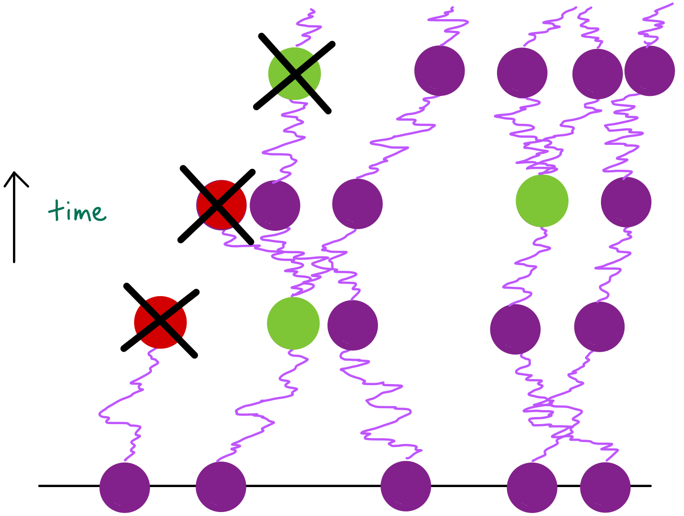


N particles in the system at all times.

Introduced by Maillard (2012). Natural continuous-time analogue of discrete-time processes introduced by Brunet and Derrida (1997).

Toy model for a population under natural selection.
Position of a particle on $\mathbb{R}$ represents evolutionary fitness.
Individuals with lowest fitness are killed.

Want to understand long-term behaviour for large N (speed + shape of cloud of particles, genealogies)

One tool: over a fixed timescale, as $N \to \infty$, density converges to solution of a free boundary problem.

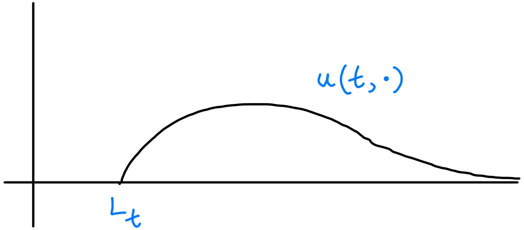
Notation $X^{(N)}(t) = (X_1^{(N)}(t), \dots, X_N^{(N)}(t))$ particle positions at time t.

$L_t^{(N)} = \min_{i \in \{1,\dots,N\}} X_i^{(N)}(t)$ leftmost particle position at time t.

Free boundary problem

Given a probability density $u_0 : \mathbb{R} \to \mathbb{R}_+$, find a pair $(u(t,x), L_t)$ that solves

$$(\text{FBP1}) \begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u & \text{for } t > 0, x > L_t \\ u(t, L_t) = 0 & \text{for } t > 0 \\ \int_{L_t}^{\infty} u(t,y)\,dy = 1 & \text{for } t > 0 \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R}. \end{cases}$$



A unique solution exists (Berestycki, Brunet, P. 2019).

It turns out that for large N,

$$u(t,x) \approx \text{density of particles at } x \text{ at time } t \approx \lim_{\delta \searrow 0} \frac{1}{N}\frac{1}{2\delta} \#\{\text{particles in } (x-\delta, x+\delta) \text{ at time } t\}$$

$$L_t \approx \text{position of leftmost particle at time } t = L_t^{(N)}.$$

Why do we get this FBP?

- For $x > L_t \approx L_t^{(N)}$, particles
  - move according to BMs
  - branch into two particles at rate 1

  $$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u$$
- At $x = L_t^{(N)} \approx L_t$, particles are killed, so $u(t, L_t) = 0$.
- Total number of particles $= N$, so $\int_{L_t}^{\infty} u(t,y)\,dy = 1$.

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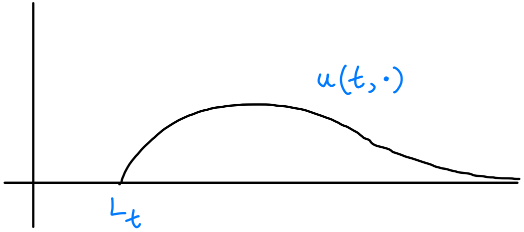

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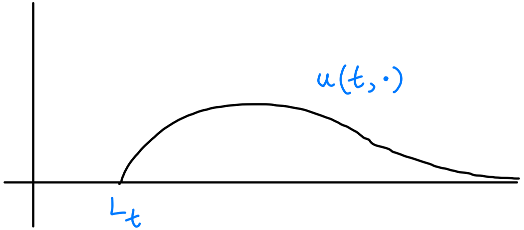

Hydrodynamic limit

Theorem (De Masi, Ferrari, Presutti, Soprano-Loto 2017) Suppose $X_1^{(N)}(0), \dots, X_N^{(N)}(0)$ are i.i.d. with density u_0. Then for $x \in \mathbb{R}$, $t > 0$,

$$\frac{1}{N} \# \{ i \leq N : X_i^{(N)}(t) \geq x \} \longrightarrow \int_x^{\infty} u(t,y)\,dy \quad \text{a.s. as } N \to \infty.$$

Earlier result: Durrett + Remenik (2011). Hydrodynamic limit of a branching-selection system in continuous time. New particles jump from location of parent after branching. Leftmost particle killed.

Main proof idea: On short time intervals, sandwich N-BBM between two processes that are easier to control.

Proof of hydrodynamic limit result

Notation

Branching Brownian motion (BBM): particles move according to independent BMs
branch into two particles at rate 1.

$X^+(t) = (X_1^+(t), \dots, X_{N_t^+}^+(t))$ locations of particles at time t.
(ordering not important - could use Ulam-Harris)

$H^+(t,x) := \frac{1}{N} \# \{ i \leq N_t^+ : X_i^+(t) \geq x \} = \frac{1}{N} \# \{$particles in BBM $\geq x$ at time $t\}$.

N-BBM. $X^{(N)}(t) = (X_1^{(N)}(t), \dots, X_N^{(N)}(t))$ locations of particles at time t.

Ordering: at time 0, particles are labelled $1, 2, \dots, N$.
When particle with label j branches, if leftmost particle has label k then the two new particles are given labels j and k (if $j = k$ then nothing happens).
Labels don't change between branching events.

$H^{(N)}(t,x) := \frac{1}{N} \# \{ i \leq N : X_i^{(N)}(t) \geq x \} = \frac{1}{N} \# \{$particles in N-BBM $\geq x$ at time $t\}$.

Proof of hydrodynamic limit result

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Lemma (upper bound coupling) For any $\mathcal{X} = (\mathcal{X}_1, \dots, \mathcal{X}_N) \in \mathbb{R}^N$, there exists a coupling of the BBM $(X^+(t), t \geq 0)$ and the N-BBM $(X^{(N)}(t), t \geq 0)$ such that under the coupling,

$$X^{(N)}(0) = \mathcal{X} = X^+(0) \qquad \text{and} \qquad H^{(N)}(t,x) \leq H^+(t,x) \quad \forall t \geq 0, \ x \in \mathbb{R}.$$

Proof: Particle system consisting of red and blue particles.
At time 0, particle configuration is given by $\mathcal{X}$, and all N particles are blue.
Particles move according to independent BMs and branch at rate 1.
When a blue particle branches, the two offspring particles are coloured blue, and the leftmost blue particle in the system is coloured red.
When a red particle branches, the two offspring particles are red.
The blue particles form an N-BBM and the whole system of particles forms a BBM.
Under this coupling,

$$\# \{ i \leq N : X_i^{(N)}(t) \geq x \} \leq \# \{ i \leq N_t^+ : X_i^+(t) \geq x \}. \ \square.$$

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Notation: For $\mathcal{X} \in \mathbb{R}^m$, $\mathcal{X}' \in \mathbb{R}^{m'}$, write $\mathcal{X} \succcurlyeq \mathcal{X}'$ iff

$|\mathcal{X} \cap [x,\infty)| \ge |\mathcal{X}' \cap [x,\infty)| \ \forall x \in \mathbb{R}$ iff $m \ge m'$ and $\exists$ permutation σ of $\{1,\dots,m\}$ s.t. $\mathcal{X}_{\sigma(i)} \ge \mathcal{X}'_i \ \forall i \le m'.$

Lemma (Lower bound coupling) Suppose $\mathcal{X} \in \mathbb{R}^N$, $\mathcal{X}^+ \in \mathbb{R}^m$ and $\mathcal{X} \succcurlyeq \mathcal{X}^+$. There exists a coupling of the N-BBM $(X^{(N)}(t), t \ge 0)$ and the BBM $(X^+(t), t \ge 0)$ such that under the coupling,

$$X^{(N)}(0) = \mathcal{X}, \quad X^+(0) = \mathcal{X}^+, \quad \text{and for } t \ge 0, \ H^{(N)}(t,x) \ge H^+(t,x) \ \forall x \in \mathbb{R} \ \text{if } N_t^+ \le N.$$

↖ # particles in BBM at time t

↗ iff $X^{(N)}(t) \succcurlyeq X^+(t)$

Proof: Let $\tau_i^+ = i^{th}$ branching time in X^+.

Claim: For $\mathcal{X} \in \mathbb{R}^N$, $\mathcal{X}^+ \in \mathbb{R}^{m_+}$ with $\mathcal{X} \succcurlyeq \mathcal{X}^+$, can couple $(X^{(N)}(t), t \ge 0)$ and $(X^+(t), t \ge 0)$ in such a way that

$$X^{(N)}(0) = \mathcal{X}, \quad X^+(0) = \mathcal{X}^+ \quad \text{and} \quad X^{(N)}(t) \succcurlyeq X^+(t) \quad \forall t \in \begin{cases} [0, \tau_1^+] & \text{if } |\mathcal{X}^+| < N \\ [0, \tau_1^+) & \text{if } |\mathcal{X}^+| = N. \end{cases}$$

Assuming the claim, get the result by applying the claim successively on time intervals $[0, \tau_1^+]$, $[\tau_1^+, \tau_2^+], \dots, [\tau_{N-m}^+, \tau_{N-m+1}^+)$

Proof of hydrodynamic limit result

Claim: For $\mathcal{X}\in\mathbb{R}^N$, $\mathcal{X}^+\in\mathbb{R}^m$ with $\mathcal{X}\succcurlyeq\mathcal{X}^+$, can couple $(X^{(N)}(t),t\geq 0)$ and $(X^+(t),t\geq 0)$ in such a way that

$$X^{(N)}(0)=\mathcal{X},\quad X^+(0)=\mathcal{X}^+ \quad\text{and}\quad X^{(N)}(t)\succcurlyeq X^+(t) \quad \forall t\in\begin{cases}[0,\tau_1^+] & \text{if } |\mathcal{X}^+|<N\\ [0,\tau_1^+) & \text{if } |\mathcal{X}^+|=N.\end{cases}$$

Proof of claim: Assume (by reordering) $\mathcal{X}_i\geq\mathcal{X}_i^+$ $\forall i\leq m$.

Let τ_i = i^{th} branching time in $X^{(N)}$

j_i = index of particle that branches at time τ_i

k_i = index of leftmost particle at time τ_i-.

Couple branching times so $\tau_1^+=\tau_{i^+}$, where $i^+=\min\{i\geq 1: j_i\leq m\}$.

Couple BMs up to time τ_1: Let $(B_i(t),t\geq 0)$ for $i\leq N$ be i.i.d. BMs starting at 0.

Let $X_i^{(N)}(t)=\mathcal{X}_i+B_i(t)$ $\quad t<\tau_1,\ i\leq N,$

$X_i^+(t)=\mathcal{X}_i^++B_i(t)$ $\quad t<\tau_1,\ i\leq m,$ so for $i\leq m$, $X_i^{(N)}(t)\geq X_i^+(t)$ $\forall t<\tau_1$

$\Longrightarrow X^{(N)}(t)\succcurlyeq X^+(t)$ $\forall t<\tau_1$.

At time τ_1, if $\tau_1\neq\tau_1^+$ (i.e. if $j_1>m$), for $i\leq N$, $X_i^{(N)}(\tau_1)=\begin{cases}X_i^{(N)}(\tau_1-) & i\neq k_1\\ X_{j_1}^{(N)}(\tau_1-) & i=k_1\end{cases}$

so for $i\leq m$, $X_i^{(N)}(\tau_1)\geq X_i^{(N)}(\tau_1-)\geq X_i^+(\tau_1-)=X_i^+(\tau_1)$. Hence $X^{(N)}(\tau_1)\succcurlyeq X^+(\tau_1)$.

Same construction on $[\tau_1,\tau_2],\dots,[\tau_{i^+-1},\tau_{i^+})$ $\Longrightarrow X_i^{(N)}(t)\geq X_i^+(t)$ for $t<\tau_1^+$, $i\leq m$

$\Longrightarrow X^{(N)}(t)\succcurlyeq X^+(t)$ for $t<\tau_1^+$. Now assume $m<N$.

At time $\tau:=\tau_1^+$, particle $j^+:=j_{i^+}$ branches in BBM and N-BBM, and particle $k^+:=k_{i^+}$ is killed in N-BBM.

$$\left.\begin{array}{ll}\text{For } k\leq m,\ k\neq k^+, & X_k^{(N)}(\tau)=X_k^{(N)}(\tau-)\geq X_k^+(\tau-)\\ \text{and} & X_{k^+}^{(N)}(\tau)=X_{j^+}^{(N)}(\tau-)\geq X_{j^+}^+(\tau-).\\ \text{If } k^+\leq m, & X_{m+1}^{(N)}(\tau)\geq X_{k^+}^{(N)}(\tau-)\geq X_{k^+}^+(\tau-).\end{array}\right\}\begin{array}{l}m+1 \text{ particles in}\\ X^+(\tau),\text{ each}\leq\text{a}\\ \text{different particle in } X^{(N)}(\tau)\end{array}$$

So $X^{(N)}(\tau_1^+)\succcurlyeq X^+(\tau_1^+)$. $\square$

Proof of hydrodynamic limit result

For $f : \mathbb{R} \to \mathbb{R}$ and $m \in \mathbb{R}$, let $C_m f(x) = \min(f(x), m)$ $x \in \mathbb{R}$. "cut"

For $t > 0$ and $f : \mathbb{R} \to \mathbb{R}$, let $G_t f(x) = \mathbb{E}_x[f(B_t)] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} f(y)\,dy$ $x \in \mathbb{R}$. "spread"

Take $c > 0$ small and $t > 0$ fixed. Take N large.

Take $\mathcal{X} \in \mathbb{R}^N$ and let $v_0^{(N)}(y) = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{x_i \geq y}$. Then $G_t v_0^{(N)}(x) = \frac{1}{N}\sum_{i=1}^{N} \mathbb{P}_x(x_i \geq B_t)$.

If $X^+(0) = \mathcal{X}$ then w.h.p., $H^+(t,x) = \frac{1}{N}\sum_{i=1}^{N} e^t \,\mathbb{P}_{x_i}(B_t \geq x) + O(N^{-c}) = e^t G_t v_0^{(N)}(x) + O(N^{-c})$ $\forall x \in \mathbb{R}$.

By upper bound coupling, if $X^{(N)}(0) = \mathcal{X}$, $H^{(N)}(t,x) \leq C_1 H^+(t,x) \leq C_1 e^t G_t v_0^{(N)}(x) + O(N^{-c})$ $\forall x \in \mathbb{R}$ w.h.p.

↖ where $X^+(0) = \mathcal{X}$

By lower bound coupling, if $X^{(N)}(0) = \mathcal{X}$, then letting $X^+(0) = \mathcal{X}^+ =$ the $N(e^{-t} - N^{-c})$ rightmost particles in $\mathcal{X}$ (so $\mathcal{X} \geq \mathcal{X}^+$ and $\frac{1}{N}\sum_{i=1}^{|\mathcal{X}^+|} \mathbb{1}_{x_i^+ \geq x} = C_{e^{-t} - N^{-c}} v_0^{(N)}(x)$)

if $N_t^+ \leq N$ then $H^{(N)}(t,x) \geq H^+(t,x) \geq e^t G_t C_{e^{-t} - N^{-c}} v_0^{(N)}(x) - O(N^{-c})$ $\forall x$ w.h.p.

↖ this happens w.h.p.

So for $\delta > 0$ small,

$$e^{\delta} G_\delta C_{e^{-\delta}} v_0^{(N)}(x) - O(N^{-c}) \leq H^{(N)}(\delta, x) \leq C_1 e^{\delta} G_\delta v_0^{(N)}(x) + O(N^{-c}) \quad \forall x \in \mathbb{R} \quad \text{w.h.p.}$$

For $t > 0$ fixed, taking $\delta \sim N^{-c'}$ s.t. $t/\delta = n \in \mathbb{N}$, by iterating,

$$(e^{\delta} G_\delta C_{e^{-\delta}})^n v_0^{(N)}(x) - O(N^{-c''}) \leq H^{(N)}(\overset{t}{\overset{\parallel}{n\delta}}, x) \leq (C_1 e^{\delta} G_\delta)^n v_0^{(N)}(x) + O(N^{-c''}) \quad \forall x \quad \text{w.h.p.}$$

Proof of hydrodynamic limit result

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For $t>0$ fixed, taking $\delta\sim N^{-c'}$ s.t. $t/\delta = n\in\mathbb{N}$,

$$(e^\delta G_\delta C_{e^{-\delta}})^n v_0^{(N)}(x) - \mathcal{O}(N^{-c''}) \le H^{(N)}(t,x) \le (C_1 e^\delta G_\delta)^n v_0^{(N)}(x) + \mathcal{O}(N^{-c''}) \quad \forall x \text{ w.h.p.}$$

Lemma Let $v(t,x) = \int_x^\infty u(t,y)\,dy$, where (u,L) solves (FBP1) with initial condition u_0.

Let $v_0(x) = \int_x^\infty u_0(y)\,dy$. Then for $n\in\mathbb{N}$ and $\delta>0$,

$$(e^\delta G_\delta C_{e^{-\delta}})^n v_0(x) \le v(n\delta, x) \le (C_1 e^\delta G_\delta)^n v_0(x) \qquad \forall x\in\mathbb{R}.$$

Cut then grow/spread.

Mass to the right of x in N-BBM with N "=" ∞. Grow/spread and cut at same time.

Grow/spread then cut.

Proof: Use Feynman-Kac formula.

Lemma For $v_0:\mathbb{R}\to[0,1]$, $\delta>0$ and $n\in\mathbb{N}$, $\|(C_1 e^\delta G_\delta)^n v_0 - (e^\delta G_\delta C_{e^{-\delta}})^n v_0\|_\infty \le (e^{n\delta}+1)(e^\delta - 1)$.

Proof:
1. $\|G_\delta f - G_\delta g\|_\infty \le \|f-g\|_\infty$
2. $\|C_1 f - C_1 g\|_\infty \le \|f-g\|_\infty$
3. If $\|f\|_\infty \le 1$ then $\|C_1 e^\delta f - f\|_\infty \le \max(e^\delta - 1, 1-e^{-\delta}) = e^\delta - 1$.
4. $e^\delta G_\delta C_{e^{-\delta}} f = G_\delta e^\delta C_{e^{-\delta}} f = G_\delta C_1 e^\delta f$.

So $\|(C_1 e^\delta G_\delta)^n v_0 - (e^\delta G_\delta C_{e^{-\delta}})^n v_0\|_\infty \le \|C_1 e^\delta G_\delta (C_1 e^\delta G_\delta)^{n-1} v_0 - G_\delta (C_1 e^\delta G_\delta)^{n-1} v_0\|_\infty$

$$+ \|G_\delta (C_1 e^\delta G_\delta)^{n-1} v_0 - \overbrace{(G_\delta C_1 e^\delta)^n v_0}^{\substack{G_\delta (C_1 e^\delta G_\delta)^{n-1} C_1 e^\delta v_0 \\ \|}}\|_\infty$$

$$\le \overset{\text{by } 1+3}{e^\delta - 1} + \overset{\text{by } 1+2}{e^{\delta(n-1)}} \|v_0 - C_1 e^\delta v_0\|_\infty \le (e^{n\delta}+1)(e^\delta - 1) \text{ by 3. } \square$$

Proof of hydrodynamic limit result

For $f:\mathbb{R}\to\mathbb{R}$ and $m\in\mathbb{R}$, let $C_m f(x) = \min(f(x), m)$ $\quad x\in\mathbb{R}$.

For $t>0$ and $f:\mathbb{R}\to\mathbb{R}$, let $G_t f(x) = \mathbb{E}_x[f(B_t)] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} f(y)\,dy$ $\quad x\in\mathbb{R}$.

For $t>0$ fixed, taking $\delta \sim N^{-c'}$ s.t. $t/\delta = n\in\mathbb{N}$,

$$(e^\delta G_\delta C_{e^{-\delta}})^n v_0^{(N)}(x) - \mathcal{O}(N^{-c''}) \le H^{(N)}(t,x) \le (C_1 e^\delta G_\delta)^n v_0^{(N)}(x) + \mathcal{O}(N^{-c''}) \quad \forall x \text{ w.h.p.}$$

Lemma Let $v(t,x)=\int_x^\infty u(t,y)\,dy$, where (u,L) solves (FBP1) with initial condition u_0.
Let $v_0(x)=\int_x^\infty u_0(y)\,dy$. Then for $n\in\mathbb{N}$ and $\delta>0$,

$$(e^\delta G_\delta C_{e^{-\delta}})^n v_0(x) \le v(n\delta,x) \le (C_1 e^\delta G_\delta)^n v_0(x) \qquad \forall x\in\mathbb{R}.$$

Lemma For $v_0:\mathbb{R}\to[0,1]$, $\delta>0$ and $n\in\mathbb{N}$, $\quad \|(C_1 e^\delta G_\delta)^n v_0 - (e^\delta G_\delta C_{e^{-\delta}})^n v_0\|_\infty \le (e^{n\delta}+1)(e^\delta-1)$

For N large, if $X_1^{(N)}(0),\dots,X_N^{(N)}(0)$ are i.i.d. with density u_0,

$$v_0^{(N)}(x) = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{X_i^{(N)}(0)\geq x} \approx \mathbb{P}(X_1^{(N)}(0)\geq x) = v_0(x) \qquad \forall x\in\mathbb{R} \quad \text{w.h.p.}$$

So w.h.p. $\forall x\in\mathbb{R}$,

$$\underset{\displaystyle v(n\delta,x)=v(t,x)}{\underbrace{(e^\delta G_\delta C_{e^{-\delta}})^n v_0(x)}_{\parallel}} - o(1) \le H^{(N)}(t,x) \le \underset{\displaystyle v(n\delta,x)=v(t,x).}{\underbrace{(C_1 e^\delta G_\delta)^n v_0(x)}_{\parallel}} + o(1)$$

Long-term behaviour of N-BBM for large N

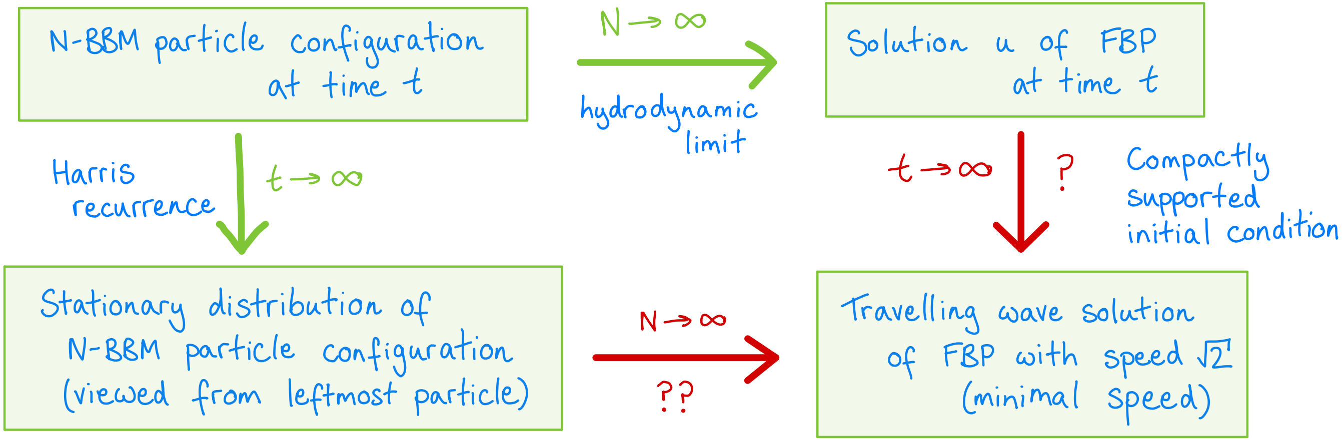
Asymptotic speed

$L_t^{(N)} = \min_{i \le N} X_i^{(N)}(t)$. $\quad \exists$ deterministic a_N s.t. $\lim_{t\to\infty} \frac{L_t^{(N)}}{t} = a_N$ a.s.

and $a_N \to \sqrt{2}$ as $N \to \infty$.

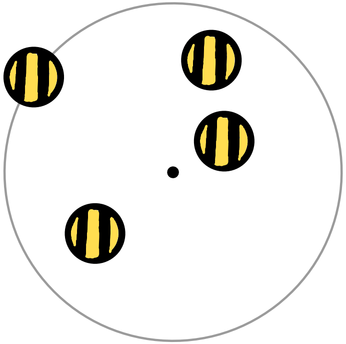
N.B. $\lim_{t\to\infty} \max_{i \le N_t^+} \frac{X_i^+(t)}{t} = \sqrt{2}$ a.s.

Selection principle



Selection principle: both PDE and particle system 'select' the same travelling wave to determine long-term behaviour.

Brownian bees



- N particles move in $\mathbb{R}^d$ according to independent Brownian motions.
- Each particle, independently, branches into two particles after an Exp(1) time.
- Each time a particle branches, the particle in the system furthest from the origin is killed.

↖ Euclidean distance

N particles in the system at all times.

Can determine long-term behaviour for large N through connection with a free boundary problem.

Notation: $X^{(N)}(t) = (X_1^{(N)}(t), \dots, X_N^{(N)}(t))$ particle positions (in $\mathbb{R}^d$) at time t.

$M_t^{(N)} = \max_{i \in \{1,\dots,N\}} \|X_i^{(N)}(t)\|$ maximum particle distance from 0 at time t.

↖ $\|\cdot\|$ is Euclidean (ℓ_2) norm

Guess: is $\lim_{N\to\infty} \lim_{t\to\infty} M_t^{(N)} = \begin{cases} 0 \\ \text{const.} \\ \infty \end{cases}$?

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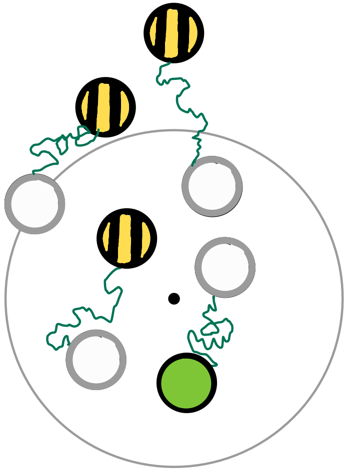

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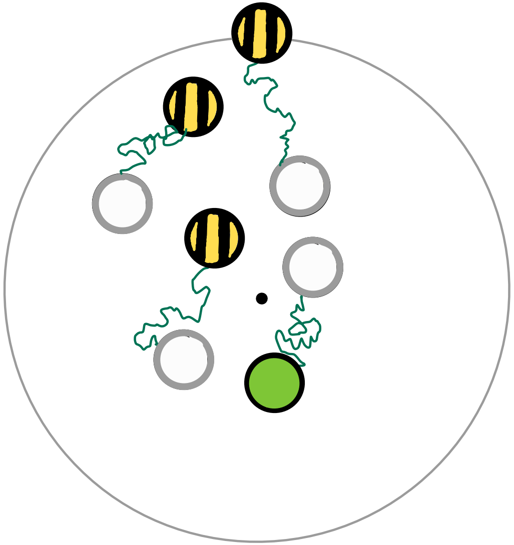
Brownian bees



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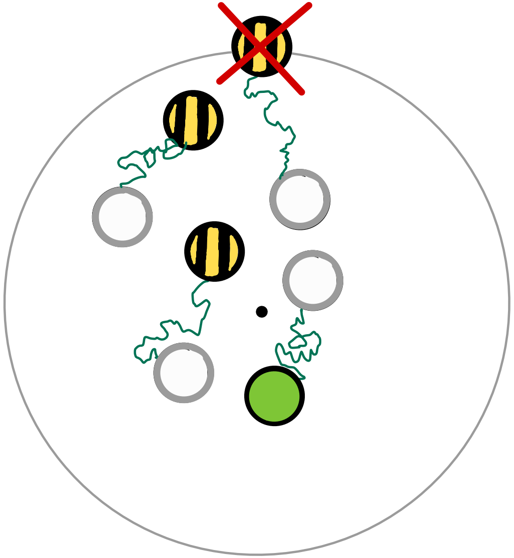

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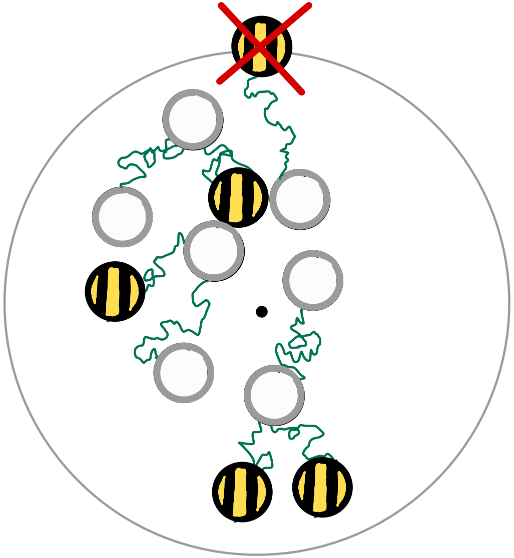
Brownian bees



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N particles in the system at all times.

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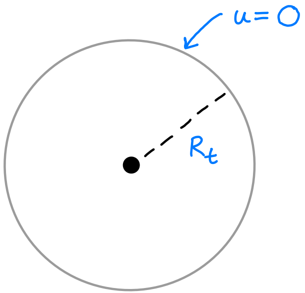
Guess: is $\lim_{N\to\infty} \lim_{t\to\infty} M_t^{(N)} = \begin{cases} 0 \\ \text{const.} \\ \infty \end{cases}$?

Free boundary problem

Given an initial probability measure μ_0 on $\mathbb{R}^d$, find a pair $(u(t,x), R_t)$ that solves

$$(\mathrm{FBP2})\begin{cases} \dfrac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u & \|x\| < R_t,\ t>0 \\ u(t,x) = 0 & \|x\| \geqslant R_t,\ t>0 \\ \displaystyle\int_{\|x\|\leq R_t} u(t,x)\,dx = 1 & t>0 \\ u(t,x)\,dx \longrightarrow \mu_0(dx) \text{ weakly as } t \searrow 0. \end{cases}$$



Theorem (Berestycki, Brunet, Nolen, P. 2020) For any Borel probability measure μ_0 on $\mathbb{R}^d$, there is a unique solution (u,R) to (FBP2). Moreover, $t \mapsto R_t$ is continuous on $(0,\infty)$.

Write $B_r(x) := \{y \in \mathbb{R}^d : \|x-y\| < r\}$.

It turns out that for large N, $u(t,x) \approx$ density of particles at x at time t

$$\text{"="} \lim_{\delta \searrow 0} \frac{1}{N} \frac{1}{\mathrm{Vol}(B_\delta(0))} \#\{\text{particles in } B_\delta(0) \text{ at time } t\}.$$

$R_t \approx$ largest particle distance from x at time t $= M_t^{(N)}$.

Why do we get this FBP? For $\|x\| < R_t \approx M_t^{(N)}$, particles move according to BMs

branch into two particles at rate 1

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u$$

At distance $R_t \approx M_t^{(N)}$ from 0, particles are killed, so $u(t,x) = 0$.

Total number of particles $= N$, so $\displaystyle\int_{\|x\|\leq R_t} u(t,x)\,dx = 1$.

Hydrodynamic limit

Notation: $\mu^{(N)}(t,dx) = \frac{1}{N}\sum_{k=1}^{N} \delta_{X_k^{(N)}(t)}(dx)$ empirical measure of particles at time t.

Theorem (BBNP) Suppose μ_0 is a Borel probability measure on $\mathbb{R}^d$, and

- $X_1^{(N)}(0), \dots, X_N^{(N)}(0)$ are i.i.d. with distribution μ_0
- (u,R) is the solution of (FBP2) with initial condition μ_0.

Then for any $t>0$ and any measurable $A \subseteq \mathbb{R}^d$, almost surely

$$\mu^{(N)}(t,A) \longrightarrow \int_A u(t,x)\,dx \quad \text{and} \quad M_t^{(N)} \longrightarrow R_t \quad \text{as } N \longrightarrow \infty.$$

Brownian bees $\mu^{(N)}(t,dx)$

$\xrightarrow[N\to\infty]{}$

Solution of FBP $(u(t,\cdot), R_t)$

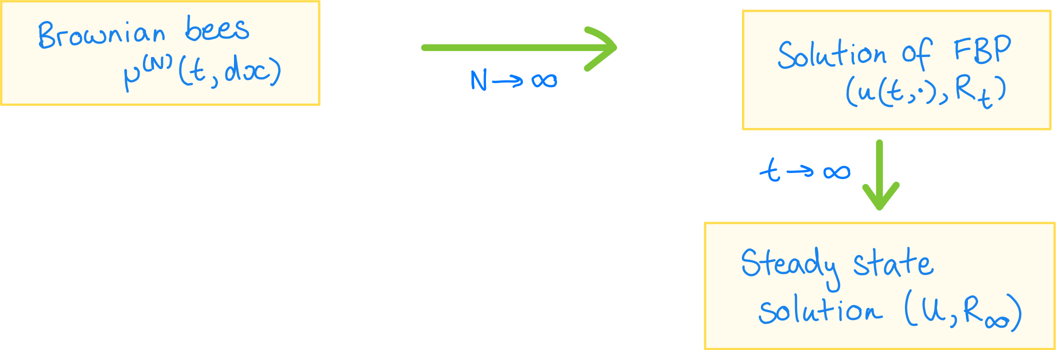
Long-term behaviour of FBP solutions

Let $(U(x), R_\infty)$ be the unique solution to

$$\begin{cases} -\Delta U(x) = U(x) & \|x\| < R_\infty \\ U(x) > 0 & \|x\| < R_\infty \\ U(x) = 0 & \|x\| \geq R_\infty \\ \displaystyle\int_{\|x\| \leq R_\infty} U(x)\,dx = 1 & \end{cases}$$

Theorem (BBNP) For any initial Borel probability measure μ_0, the solution (u, R) of (FBP2) satisfies

$$\lim_{t\to\infty} R_t = R_\infty \quad \text{and} \quad \lim_{t\to\infty} \| u(t,\cdot) - U(\cdot) \|_\infty = 0.$$



Brownian bees $\mu^{(N)}(t,dx)$

$N \to \infty$

Solution of FBP $(u(t,\cdot), R_t)$

$t \to \infty$

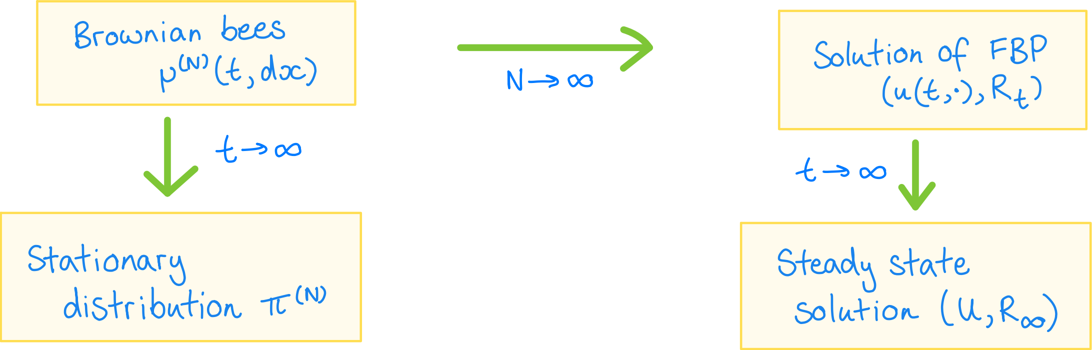
Steady state solution (U, R_∞)

Stationary distribution

Theorem (BBNP) The process $(X^{(N)}(t), t \geq 0)$ has a unique invariant measure $\pi^{(N)}$, which is a probability measure on $(\mathbb{R}^d)^N$.

For any initial particle configuration, the law of $X^{(N)}(t)$ converges in total variation norm to $\pi^{(N)}$ as $t \to \infty$. In particular, for $C \subseteq (\mathbb{R}^d)^N$ measurable,

$$\mathbb{P}(X^{(N)}(t) \in C) \to \pi^{(N)}(C) \quad \text{as } t \to \infty.$$



Brownian bees $\mu^{(N)}(t, dx)$

$N \to \infty$

Solution of FBP $(u(t,\cdot), R_t)$

$t \to \infty$

$t \to \infty$

Stationary distribution $\pi^{(N)}$

Steady state solution (U, R_∞)

Selection principle

Theorem (BBNP) For $\varepsilon>0$ and $A\subseteq\mathbb{R}^d$ measurable, as $N\to\infty$,

$$\pi^{(N)}\left(\left\{X\in(\mathbb{R}^d)^N : \left|\frac{1}{N}\sum_{i=1}^{N}\mathbb{1}_{X_i\in A}-\int_A u(x)\,dx\right|\geq\varepsilon\right\}\right)\to 0$$

and $$\pi^{(N)}\left(\left\{X\in(\mathbb{R}^d)^N : \left|\max_{i\in\{1,\dots,N\}}\|X_i\|-R_\infty\right|\geq\varepsilon\right\}\right)\to 0.$$

i.e. for N large, for $(X_1,\dots,X_N)\in(\mathbb{R}^d)^N$ with law $\pi^{(N)}$,

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{1}_{X_i\in A}\approx\int_A u(x)\,dx \quad\text{and}\quad \max_{i\in\{1,\dots,N\}}\|X_i\|\approx R_\infty \quad\text{w.h.p.}$$

$$\begin{tikzcd}
\boxed{\text{Brownian bees } \mu^{(N)}(t,dx)} \arrow[r, "N\to\infty"'] \arrow[d, "t\to\infty"] & \boxed{\text{Solution of FBP } (u(t,\cdot),R_t)} \arrow[d, "t\to\infty"'] \\
\boxed{\text{Stationary distribution } \pi^{(N)}} \arrow[r, "N\to\infty"] & \boxed{\text{Steady state solution } (u,R_\infty)}
\end{tikzcd}$$

Proof of hydrodynamic limit

Let $F^{(N)}(t,r) := \mu^{(N)}(B_r(0),t) = \frac{1}{N}\#\{i\in\{1,\dots,N\}: \|X_i^{(N)}(t)\| < r\}$.

One-dimensional free boundary problem

Given $v_0 : [0,\infty) \to [0,1]$ measurable, find a pair $(v(t,r), R_t)$ such that

$$(\text{FBP3}) \begin{cases} \dfrac{\partial v}{\partial t} = \frac{1}{2}\Delta v - \dfrac{d-1}{2r}\dfrac{\partial v}{\partial r} + v & t>0,\ r\in(0,R_t) \\ v(t,r) = 1 & t>0,\ r\geq R_t \\ \dfrac{\partial v}{\partial r}(t,R_t) = 0 & t>0 \\ v(t,0) = 0 & t>0 \\ v(0,\cdot) = v_0 \end{cases}$$

For N large, $v(t,r) \approx F^{(N)}(t,r)$.

Proposition For any $v_0 : [0,\infty) \to [0,1]$ measurable, there exists a unique solution (v,R) to (FBP3).

Proposition Let $v_0(r) = \mu_0(B_r(0))$. Suppose (v,R) solves (FBP3) with initial condition v_0, and $(u,\tilde{R})$ solves (FBP2) with initial condition μ_0. Then for $t>0$ and $r\geq 0$,

$$R_t = \tilde{R}_t \quad \text{and} \quad \underset{\substack{\nearrow \\ \text{proportion within distance } r \text{ of } 0}}{v(t,r)} = \int_{\|x\|<r} \underset{\substack{\nwarrow \\ \text{density at } x}}{u(t,x)}\,dx.$$

Proof of hydrodynamic limit

Let $F^{(N)}(t,r) := \mu^{(N)}(B_r(0),t) = \frac{1}{N}\#\{i\in\{1,\dots,N\} : \|X_i^{(N)}(t)\| < r\}$.

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(\text{FBP3}) \begin{cases}
\frac{\partial v}{\partial t} = \frac{1}{2}\Delta v - \frac{d-1}{2r}\frac{\partial v}{\partial r} + v & t>0,\ r\in(0,R_t) \\
v(t,r) = 1 & t>0,\ r\geq R_t \\
\frac{\partial v}{\partial r}(t,R_t) = 0 & t>0 \\
v(t,0) = 0 & t>0 \\
v(0,\cdot) = v_0 &
\end{cases}
$$

Notation: For $x^{(N)} \in (\mathbb{R}^d)^N$, write $\mathbb{P}_{x^{(N)}}(\cdot) = \mathbb{P}(\cdot \mid X^{(N)}(0) = x^{(N)})$.

Proposition (one-dimensional hydrodynamic limit) There exists $c_1 > 0$ such that for N sufficiently large, for $t>0$ and $x^{(N)} \in (\mathbb{R}^d)^N$,

$$\mathbb{P}_{x^{(N)}}\left(\sup_{r\geq 0}|F^{(N)}(t,r) - v^{(N)}(t,r)| \geq e^{2t}N^{-c_1}\right) \leq e^t N^{-1-c_1},$$

where $(v^{(N)}, R^{(N)})$ solves (FBP3) with $v_0(r) = F^{(N)}(0,r)$. $\leftarrow$ so under $\mathbb{P}_{x^{(N)}}$,

$$v_0(r) = \frac{1}{N}\sum_{i=1}^{N}\mathbb{1}_{\|x_i^{(N)}\| < r}$$

Proof: Similar to N-BBM – upper and lower bound couplings.

Proof of hydrodynamic limit

Coupling with d-dimensional BBM: $(X_i^+(t), i \leq N_t^+)$ particle positions at time t in BBM.

For $0 \leq s \leq t$ and $i \leq N_t^+$, $X_{i,t}^+(s) :=$ position of time-s ancestor of particle labelled i at time t.

Couple so $\forall t$ $X^{(N)}(t) \subseteq X^+(t)$, and

$$X^{(N)}(t) = \{ X_i^+(t) : i \leq N_t^+, \ \| X_{i,t}^+(s) \| \leq M_s^{(N)} \ \forall s \in [0,t] \}.$$

Assumptions for d-dimensional hydrodynamic limit result:

Suppose μ_0 is a Borel probability measure on $\mathbb{R}^d$, and

- $X_1^{(N)}(0), \ldots, X_N^{(N)}(0)$ are i.i.d. with distribution μ_0
- (u, R) solves (FBP2) with initial condition μ_0.

Let (v, R) solve (FBP3) with initial condition $v_0(r) = \mu_0(B_r(0))$.

Proposition There exists $c_2 > 0$ such that for any $0 < \eta < T$, for N sufficiently large,

$$\mathbb{P}(\exists t \in [\eta, T] : M_t^{(N)} > R_t + \eta) \leq N^{-1-c_2}.$$

Proof: Step 1 $\exists c_3 > 0$ s.t. for N sufficiently large, for $t \geq 0$,

$$\mathbb{P}(\| F^{(N)}(t, \cdot) - v(t, \cdot) \|_\infty \geq e^{2t} N^{-c_3}) \leq e^t N^{-1-c_3}.$$

Proof of hydrodynamic limit

Suppose μ_0 is a Borel probability measure on $\mathbb{R}^d$, and

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Proof: Step 1 $\exists c_3 > 0$ s.t. for N sufficiently large, for $t \geq 0$,

$$\mathbb{P}\left(\|F^{(N)}(t,\cdot) - v(t,\cdot)\|_\infty \geq e^{2t} N^{-c_3}\right) \le e^t N^{-1-c_3}.$$

$\Longrightarrow$ $F^{(N)}(t, R_t) \approx v(t, R_t) = 1$ w.h.p.

Proof of step 1:

Let $(v^{(N)}, R^{(N)})$ solve (FBP3) with initial condition $v_0^{(N)}(r) = F^{(N)}(0, r)$. Then

$$\|F^{(N)}(t,\cdot) - v(t,\cdot)\|_\infty \le \|F^{(N)}(t,\cdot) - v^{(N)}(t,\cdot)\|_\infty + \|v^{(N)}(t,\cdot) - v(t,\cdot)\|_\infty.$$

use one-dimensional hydrodynamic limit

use that $\|v^{(N)}(t,\cdot) - v(t,\cdot)\|_\infty \le e^t \|v_0^{(N)} - v_0\|_\infty$

$$= e^t \sup_{r \geq 0} |F^{(N)}(0,r) - \mu_0(B_r(0))|$$

+ quantitative Glivenko-Cantelli theorem.

Proof of hydrodynamic limit

Suppose μ_0 is a Borel probability measure on $\mathbb{R}^d$, and

- $X_1^{(N)}(0), \dots, X_N^{(N)}(0)$ are i.i.d. with distribution μ_0
- (u, R) solves (FBP2) with initial condition μ_0.

Let (v, R) solve (FBP3) with initial condition $v_0(r) = \mu_0(B_r(0))$.

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$$\mathbb{P}(\exists t \in [\eta, T] : M_t^{(N)} > R_t + \eta) \leq N^{-1-c_2}.$$

Proof: Step 1 $\exists c_3 > 0$ s.t. for N sufficiently large, for $t \geq 0$,

$$\mathbb{P}(\| F^{(N)}(t,\cdot) - v(t,\cdot) \|_\infty \geq e^{2t} N^{-c_3}) \leq e^t N^{-1-c_3}.$$

$\Longrightarrow$ $F^{(N)}(t, R_t) \approx v(t, R_t)$ w.h.p. $= 1$

Step 2 Let $\varepsilon = N^{-c_3/2}$. For N sufficiently large, for $t \in [0,T]$,

$$\mathbb{P}(\exists s \in [\varepsilon, 2\varepsilon] : M_{t+s}^{(N)} > R_t + \varepsilon^{1/3}) \leq 2e^T N^{-1-c_3}.$$

Proof of step 2: Suppose $F^{(N)}(t, R_t) \geq 1 - e^{2t} N^{-c_3}$ (happens w.h.p. by Step 1). Then w.h.p.

- by time $t + \varepsilon$, the particles in $B_{R_t}(0)$ at time 0 have $> N$ descendants in the BBM
- on the time interval $[t, t+2\varepsilon]$, no particles in the BBM move more than distance $\frac{1}{3}\varepsilon^{1/3}$ from their time-t ancestor's position.

$\Longrightarrow$ $M_{t+s^*}^{(N)} \leq R_t + \frac{1}{3}\varepsilon^{1/3}$ some $s^* \in [0, \varepsilon]$

Proof of hydrodynamic limit

Suppose μ_0 is a Borel probability measure on $\mathbb{R}^d$, and

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Let (v, R) solve (FBP3) with initial condition $v_0(r) = \mu_0(B_r(0))$.

Proposition There exists $c_2 > 0$ such that for any $0 < \eta < T$, for N sufficiently large,

$$\mathbb{P}(\exists t \in [\eta, T] : M_t^{(N)} > R_t + \eta) \leq N^{-1-c_2}.$$

Proof: Step 1 $\exists c_3 > 0$ s.t. for N sufficiently large, for $t \geq 0$,

$$\mathbb{P}(\| F^{(N)}(t, \cdot) - v(t, \cdot) \|_\infty \geq e^{2t} N^{-c_3}) \leq e^t N^{-1-c_3}.$$

$\Longrightarrow$ $F^{(N)}(t, R_t) \approx v(t, R_t) = 1$ w.h.p.

Step 2 Let $\varepsilon = N^{-c_3/2}$. For N sufficiently large, for $t \in [0, T]$,

$$\mathbb{P}(\exists s \in [\varepsilon, 2\varepsilon] : M_{t+s}^{(N)} > R_t + \varepsilon^{1/3}) \leq 2e^T N^{-1-c_3}.$$

Apply Step 1 with $t = k\varepsilon$, for $k \in \mathbb{N}_0$, $k \leq \lfloor T/\varepsilon \rfloor$.

Since $t \mapsto R_t$ is continuous on $(0, \infty)$, we can take N sufficiently large that

$$R_{\varepsilon(\lfloor t/\varepsilon \rfloor - 1)} + \varepsilon^{1/3} \leq R_t + \eta \qquad \forall t \in [\eta, T].$$

Proof of hydrodynamic limit

Proposition There exists $c_2>0$ such that for any $0<\eta<T$, for N sufficiently large,

$$\mathbb{P}\left(\exists\, t\in[\eta,T]: M_t^{(N)} > R_t+\eta\right) \le N^{-1-c_2}.$$

Proof of d-dimensional hydrodynamic limit:

Claim $\exists\, c_4>0$ s.t. for $t>0$, $\delta>0$ and $A\subseteq\mathbb{R}^d$ measurable, for N sufficiently large,

$$\mathbb{P}\left(\mu^{(N)}(t,A)-\int_A u(t,x)\,dx \geq \delta\right) \le N^{-1-c_4}.$$

Then since $\mu^{(N)}(t,A) = 1-\mu^{(N)}(t,\mathbb{R}^d\setminus A)$ and $\int_A u(t,x)\,dx = 1-\int_{\mathbb{R}^d\setminus A} u(t,x)\,dx$,

$$\mathbb{P}\left(\mu^{(N)}(t,A)-\int_A u(t,x)\,dx \le -\delta\right) = \mathbb{P}\left(\mu^{(N)}(t,\mathbb{R}^d\setminus A)-\int_{\mathbb{R}^d\setminus A} u(t,x)\,dx \geq \delta\right) \le N^{-1-c_4}$$

for N sufficiently large, by Claim. So by Borel-Cantelli, a.s.

$$\left|\mu^{(N)}(t,A)-\int_A u(t,x)\,dx\right| < \delta \quad \text{and} \quad M_t^{(N)} > R_t-\delta' \quad \text{for } N \text{ sufficiently large.}$$

Proof of hydrodynamic limit

Proposition There exists $c_2>0$ such that for any $0<\eta<T$, for N sufficiently large,

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Proof of d-dimensional hydrodynamic limit:

Claim $\exists c_4>0$ s.t. for $t>0$, $\delta>0$ and $A\subseteq\mathbb{R}^d$ measurable, for N sufficiently large,

$$\mathbb{P}\left(\rho^{(N)}(t,A)-\int_A u(t,x)\,dx \geqslant \delta\right) \le N^{-1-c_4}.$$

Proof of claim. For $\eta>0$ and $t\geqslant 0$, let $\mathcal{C}_{\eta,t}=\{X_i^+(t): i\le N_t^+,\ \|X_{i,t}^+(s)\|\le R_s+\eta\ \forall s\in[\eta,t]\}$.

If $M_s^{(N)}\le R_s+\eta$ $\forall s\in[\eta,t]$ then $X^{(N)}(t)\subseteq\mathcal{C}_{\eta,t}$.

So $\mathbb{P}\left(\rho^{(N)}(t,A)-\int_A u(t,x)\,dx\geqslant\delta\right)$

$$\le \mathbb{P}\left(\exists s\in[\eta,t]:\ M_s^{(N)}>R_s+\eta\right)+\mathbb{P}\left(\frac{1}{N}|\mathcal{C}_{\eta,t}\cap A|-\int_A u(t,x)\,dx\geqslant\delta\right)$$

use Proposition

use $\lim_{\eta\downarrow 0}\mathbb{E}\left[\frac{1}{N}|\mathcal{C}_{\eta,t}\cap A|\right]=\int_A u(t,x)\,dx$

and $\mathbb{E}\left[\left(\frac{1}{N}|\mathcal{C}_{\eta,t}\cap A|-\mathbb{E}\left[\frac{1}{N}|\mathcal{C}_{\eta,t}\cap A|\right]\right)^4\right]\lesssim N^{-2}.$

Long-term behaviour - free boundary problem

Recall (u, R_∞) is steady state solution of (FBP2).

Let $V(r) = \int_{\|x\|<r} u(x)\,dx$. Then (V, R_∞) is a steady state solution of (FBP3).

Proposition For $c>0$, $K>0$ and $\varepsilon>0$, there exists $t_\varepsilon = t_\varepsilon(c,K) \in (0,\infty)$ such that if $v_0 : [0,\infty) \to [0,1]$ is non-decreasing with $v_0(K) \geq c$ and (v,R) solves (FBP3) with initial condition v_0 then

$$|v(t,r) - V(r)| < \varepsilon \quad \forall r \geq 0 \qquad \text{and} \qquad |R_t - R_\infty| < \varepsilon \qquad \forall t \geq t_\varepsilon \qquad (*)$$

Proof: Step 1 Show $(*)$ holds for (v^-, R^-) and (v^+, R^+), where

- (v^-, R^-) solves (FBP3) with initial condition $v_0^-(r) = c\mathbb{1}_{r \geq K}$
- (v^+, R^+) solves (FBP3) with initial condition $v_0^+(r) = 1$.

Step 2 Comparison principle: If $v_0^{(1)}(r) \leq v_0^{(2)}(r)$ $\forall r$ and $(v^{(i)}, R^{(i)})$ solves (FBP3) with initial condition $v_0^{(i)}$ $(i=1,2)$ then

$$v^{(1)}(t,r) \leq v^{(2)}(t,r) \quad \forall t>0, r\geq 0 \qquad \Rightarrow \quad R_t^{(1)} \geq R_t^{(2)} \quad \forall t>0.$$

For $v_0 : [0,\infty) \to [0,1]$ non-decreasing with $v_0(K) \geq c$, $v_0^-(r) \leq v_0(r) \leq v_0^+(r)$ $\forall r \geq 0$, so $v^-(t,r) \leq v(t,r) \leq v^+(t,r)$ $\forall t>0, r\geq 0$ and $R_t^+ \leq R_t \leq R_t^-$ $\forall t>0$.

Long-term behaviour - Brownian bees

Theorem Take $K>0$ and $c>0$. For $\varepsilon>0$, for $N \geq N_\varepsilon$ and $t \geq T_\varepsilon$, for an initial condition $x^{(N)} \in (\mathbb{R}^d)^N$ such that $F^{(N)}(0,K) \geq c$,

$$\mathbb{P}_{x^{(N)}}\Big(\sup_{r \geq 0} |F^{(N)}(t,r) - V(r)| \geq \varepsilon \Big) < \varepsilon$$

and

$$\mathbb{P}_{x^{(N)}}\big(|M_t^{(N)} - R_\infty| \geq \varepsilon \big) < \varepsilon.$$

Proof. Assume wlog K is large and c is small. Fix T large, and take $t \geq T$.

Suppose $F^{(N)}(t-T,K) \geq c$. Then

1. If N is large, w.h.p. $F^{(N)}(t,\cdot) \approx v^{(N)}(T,\cdot)$,
   where $(v^{(N)}, R^{(N)})$ solves (FBP3) with $v_0(r) = F^{(N)}(t-T,r)$,
   by 1d hydrodynamic limit result.

2. If T is large, $v^{(N)}(T,\cdot) \approx V(\cdot)$ because $v_0(K) = F^{(N)}(t-T,K) \geq c$.

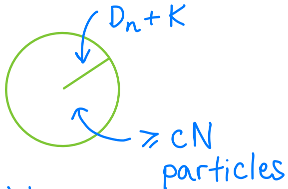
So w.h.p. $F^{(N)}(t,\cdot) \approx V(\cdot)$. Hence STP:

Claim For large s, $F^{(N)}(s,K) \geq c$ w.h.p.

Long-term behaviour – Brownian bees

Claim For large s, $F^{(N)}(s,K) \geqslant c$ w.h.p. (Assuming K is large and c is small.)

Proof of claim: Fix $t_0>0$. Let $D_n = \min\{i \in \mathbb{N}_0 : F^{(N)}(nt_0, K+i) \geqslant c\}$.



Lemma For $n \in \mathbb{N}_0$ and $m>0$, if $D_n \leqslant m$ then

$$\mathbb{P}(D_{n+1} > m+j \mid \underset{\uparrow \text{ natural filtration}}{\mathcal{F}_{nt_0}}) \leqslant 4dNe^{t_0}e^{-j^2/36dt_0} \quad \forall j \in \mathbb{N}.$$

Proof: Take $n=0$ wlog.

Coupling with BBM. Suppose $\|X_i^+(t) - X_{i,t}^+(0)\| < \frac{1}{3}j \;\; \forall t \in [0,t_0],\ i \leqslant N_t^+$.

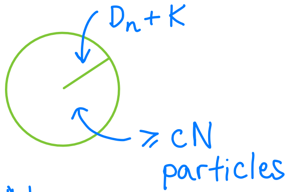
- Case 1: If $M_t^{(N)} > K+m+\frac{1}{3}j \;\; \forall t \leqslant t_0$, then no descendants of particles in $B_{K+m}(0)$ are killed by time t_0, so there are $\geqslant cN$ particles in $B_{K+m+1/3j}(0)$ at time t_0.
- Case 2: If $M_{t^*}^{(N)} \leqslant K+m+\frac{1}{3}j$ for some $t^* \leqslant t_0$, then at time t^*, all surviving particles are in $B_{K+m+1/3j}(0)$, and for $i \leqslant N_{t_0}^+$,

$$\|X_i^+(t_0) - X_{i,t_0}^+(t^*)\| \leqslant \|X_i^+(t_0) - X_{i,t_0}^+(0)\| + \|X_{i,t_0}^+(0) - X_{i,t_0}^+(t^*)\| \leqslant 2/3\, j,$$

so all surviving particles are in $B_{K+m+j}(0)$ at time t_0.

Long-term behaviour - Brownian bees

Claim For large s, $F^{(N)}(s,K) \geq c$ w.h.p. (Assuming K is large and c is small.)



Proof of claim: Fix $t_0 > 0$. Let $D_n = \min\{i \in \mathbb{N}_0 : F^{(N)}(nt_0, K+i) \geq c\}$.

Lemma For $n \in \mathbb{N}_0$ and $m > 0$, if $D_n \leq m$ then

$$\mathbb{P}(D_{n+1} > m+j \mid \mathcal{F}_{nt_0}) \leq 4dNe^{t_0}e^{-j^2/36dt_0} \quad \forall j \in \mathbb{N}.$$

↑ natural filtration

For t_0 large, for $m \geq 0$, if $v_0(K+m) \geq c$ then if (v,R) solves (FBP3),

$$v(t_0, K+m-1) \geq 2c.$$

So by 1d hydrodynamic limit, if $D_n \leq m$, $\mathbb{P}(D_{n+1} > m-1 \mid \mathcal{F}_{nt_0}) \ll N^{-1}$.

Use Lemma for $j \geq (\log N)^{2/3}$.

Can couple $(D_n)_{n=0}^{\infty}$ with a positive recurrent Markov chain $(Y_n)_{n=0}^{\infty}$ in such a way that $D_n \leq Y_n$ $\forall n$ and $\mathbb{P}(Y_n = 0) \geq 1-\varepsilon$ for n large.

Then $Y_n = 0 \Rightarrow D_n = 0 \Rightarrow F^{(N)}(nt_0, K) \geq c$.

Barycentric Brownian bees L. Addario-Berry, J. Lin, T. Tendron 2020

N particles in $\mathbb{R}^d$ moving according to Brownian motions and branching at rate 1.

Each time a particle branches, the particle furthest from the centre of mass (barycentre) of the system is killed.

Notation: $X^{(N)}(t) = (X_1^{(N)}(t), \dots, X_N^{(N)}(t))$ particle positions at time t.

$\overline{X}(t) = \frac{1}{N} \sum_{k=1}^{N} X_k^{(N)}(t)$ barycentre at time t.

Theorem (Invariance principle) For $N \geq 1$, $\exists\, \sigma = \sigma(d,N) \in (0,\infty)$ s.t. as $m \to \infty$,

$$(m^{-1/2}\, \overline{X}(tm),\ 0 \leq t \leq 1) \xrightarrow{d} (\sigma B_t)_{0 \leq t \leq 1}$$

w.r.t. Skorohod topology.

Conjecture For large N, at a large time t, for $A \subseteq \mathbb{R}^d$,

$$\frac{1}{N} \# \{ i \leq N : X_i^{(N)}(t) - \overline{X}(t) \in A \} \approx \int_A u(x)\, dx \qquad \text{w.h.p.}$$

General d-dimensional N-BBM   N. Berestycki, L. Z. Zhao 2018

$\mathcal{F}: \mathbb{R}^d \to \mathbb{R}$ fitness function. Fitness value of a particle at x is $\mathcal{F}(x)$.

N particles in $\mathbb{R}^d$ moving according to Brownian motions and branching at rate 1.

Each time a particle branches, the particle in the system with the lowest fitness value is killed.

Case $\mathcal{F}(x) = \|x\|$ Particles form a clump that moves away from 0 at a deterministic speed in a random direction.

Case $\mathcal{F}(x) = \langle \lambda, x \rangle$ Particles form a clump that moves in direction λ at a deterministic speed.

Conjecture Hydrodynamic limit. Let $\Omega_\ell = \{x \in \mathbb{R}^d : \mathcal{F}(x) > \ell\}$.

Find $(u(t,x), \ell(t))$ s.t.

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u \qquad t>0,\ x \in \Omega_{\ell(t)}$$

$$u(t,x) = 0 \qquad t>0,\ x \notin \Omega_{\ell(t)}$$

$$\int_{\mathbb{R}^d} u(t,x)\,dx = 1 \qquad t>0$$

$$u(t,\cdot) \to \mu_0 \text{ weakly as } t \searrow 0$$

N-particle branching random walk (N-BRW)

Let X be a real-valued random variable (jump distribution)

N particles with locations in $\mathbb{R}$.

At each time $n \in \mathbb{N}_0$, each particle has two offspring.

Each of the $2N$ offspring particles makes an independent jump from its parent's location, with the same law as X.

The N rightmost particles (of the $2N$ offspring particles) form the population at time $n+1$.

Notation: $X_1^{(N)}(n) \leq X_2^{(N)}(n) \leq \ldots \leq X_N^{(N)}(n)$ ordered particle positions at time n.

Asymptotic speed

If $\mathbb{E}[X] < \infty$ then $\exists\, v_N \in (0,\infty)$ s.t. $\lim_{n\to\infty} \frac{X_N^{(N)}(n)}{n} = v_N = \lim_{n\to\infty} \frac{X_1^{(N)}(n)}{n}$ a.s. and in L^1.

Theorem (Bérard and Gouéré 2010) If $\mathbb{E}[e^{\lambda X}] < \infty$ for some $\lambda > 0$ (+ technical assumptions) then $\lim_{N\to\infty} v_N = v_\infty$ exists and $v_\infty - v_N \sim c(\log N)^{-2}$ as $N \to \infty$.

Conjectured by Brunet + Derrida 1997. Related result for Fisher-KPP equation with noise (Mueller, Mytnik, Quastel 2009)

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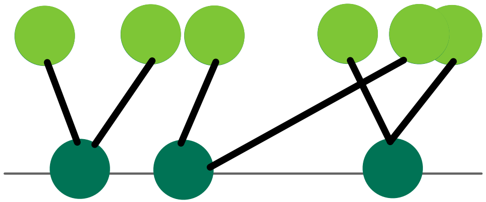

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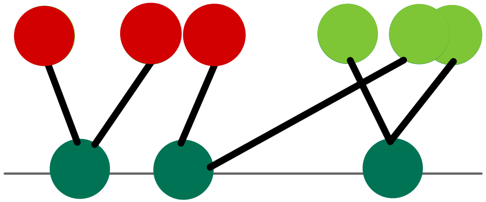

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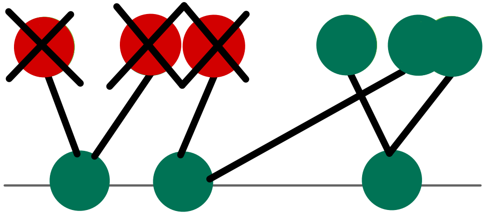

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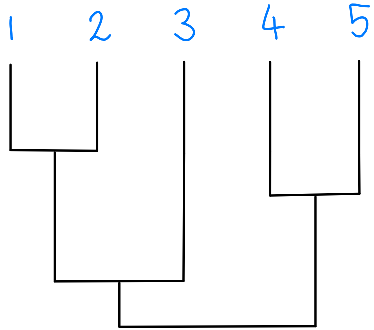
Genealogy

Fix $k \in \mathbb{N}$. Sample k particles uniformly at random from the N particles at a large time t. Trace their ancestry backwards in time

$\rightarrow$ process $(\mathcal{P}_n)_{n=0}^{\infty}$ of partitions of $\{1, \dots, k\}$ Coalescent process

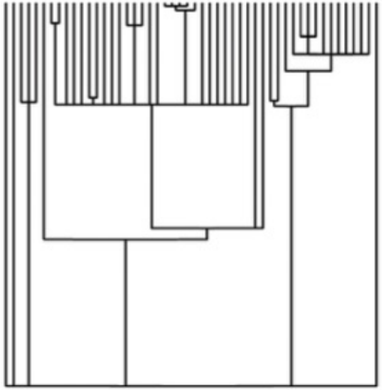
i and j in same block in $\mathcal{P}_n$ if common ancestor at time $t-n$.



Bolthausen-Sznitman coalescent

Merger rate of any given k-tuple of blocks when b blocks in total

$$= \frac{(k-2)!\,(b-k)!}{(b-1)!}$$



Thanks to A. Wakolbinger and G. Kersting

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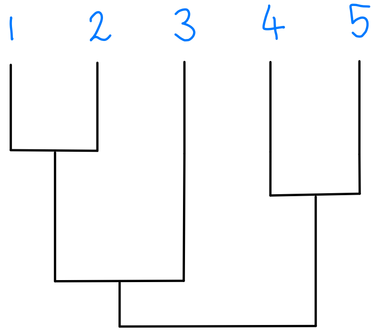

Bolthausen-Sznitman coalescent

Merger rate of any given k-tuple of blocks when b blocks in total $= \dfrac{(k-2)!\,(b-k)!}{(b-1)!}$

Conjecture (Brunet, Derrida, Mueller, Munier)

If X has exponential moments then the genealogy of a sample on a $(\log N)^3$ timescale converges to a Bolthausen-Sznitman coalescent as $N \to \infty$. (Also for N-BBM)

Berestycki, Berestycki, Schweinsberg: BBM with drift $-\sqrt{2 - \dfrac{2\pi^2}{(\log N + 3\log\log N}}$, particles killed if hit 0. Under suitable initial conditions, population has size $\sim N$.

Theorem Sample particles at time $t(\log N)^3$. After rescaling time by $\dfrac{2\pi}{(\log N)^3}$, genealogy converges to Bolthausen-Sznitman coalescent as $N \to \infty$.

N-BRW with heavy-tailed jump distribution

Suppose $\mathbb{P}(X > x) \sim x^{-\alpha}$ as $x \to \infty$.

Asymptotic speed

Theorem (Bérard and Maillard 2014)

If $\mathbb{E}[X] < \infty$, $\lim\limits_{n \to \infty} \dfrac{X_N^{(N)}(n)}{n} = v_N$ where $v_N \sim c_\alpha N^{1/\alpha} (\log N)^{1/\alpha - 1}$ as $N \to \infty$.

If $\mathbb{E}[X] = \infty$, cloud of particles accelerates.

Genealogy

Theorem (P., Roberts, Talyigás 2021)

Sample k particles at a time $t \geq 4 \log_2 N$.

The genealogy on a $\log N$ timescale is approximately given by a star-shaped coalescent when N is large.