Design Theory and Extremal Combinatorics

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Design Theory

Definition Given an integer $m \geq 1$, let $[m] = \{1, 2, \ldots, m\}$.
Definition Given integers $k \leq m$, let $\binom{[m]}{k}$ denote all k- subsets of $[m]$.

Definition Given parameters t, m, k, λ, a t-(m, k, λ) design $\mathcal{D}$ is a multiset of subsets in $\binom{[m]}{k}$ such that for each $S \in \binom{[m]}{t}$ there are exactly λ blocks $B \in \mathcal{D}$ containing S.

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Definition Given parameters t, m, k, λ, a t-(m, k, λ) packing $\mathcal{P}$ is a set of subsets in $\binom{[m]}{k}$ such that for each $S \in \binom{[m]}{t}$ there are at most λ blocks $B \in \mathcal{P}$ containing S.
(we will require a simple packing).

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Let $T_{m,\lambda}$ denote the element-triple incidence matrix of a simple 2-$S(m, 3, \lambda)$design.
Thus $T_{m,\lambda}$ is an $m \times \frac{\lambda}{3}\binom{m}{2}$ simple matrix with all columns of column sum 3 and having no submatrix

$$\overbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{bmatrix}}^{\lambda+1}$$

Definition We say that a matrix A is *simple* if it is a (0,1)-matrix with no repeated columns.

Definition Let $\mathbf{1}_k$ denote the column of k 1's.

Definition Let $\mathbf{1}_k\mathbf{0}_\ell$ denote the column of k 1's on top of ℓ 0's.

Definition Let $s \cdot F$ denote $[\overbrace{F|F|\cdots|F}^{s}]$.

Definition Let K_k^ℓ denote the simple $k \times \binom{k}{\ell}$ matrix of all columns of sum ℓ.

Theorem Let A be an $m \times n$ simple matrix with no submatrix

$$q \cdot \mathbf{1}_2 = \begin{bmatrix} \overbrace{1 \quad 1 \quad \cdots \quad 1}^{q} \\ 1 \quad 1 \quad \cdots \quad 1 \end{bmatrix}$$

Then

$$n \le \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \frac{q-2}{3}\binom{m}{2}$$

with equality only for

$$A = [K_m^0 K_m^1 K_m^2 T_{m,q-2}]$$

if $m \ge q$ and $m \equiv 1, 3 (\bmod 6)$.
Note that a $t - (m, k, \lambda)$ design has the maximum number of columns all of sum k with no submatrix $(\lambda + 1) \cdot \mathbf{1}_t$.

Theorem (A., Barekat) Let q be given. Then for $m > q$, if A is an $m \times n$ simple matrix with no submatrix which is a row permutation of

$$q \cdot \mathbf{1}_2\mathbf{0}_1 = \overbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}}^{q}$$

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$$n \le \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \frac{q-2}{3}\binom{m}{2} + \binom{m}{m}$$

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if $m \equiv 1, 3 (\text{mod } 6)$.

Theorem (A., Barekat) Let q be given. Then there exists an M so that for $m > M$, if A is an $m \times n$ simple matrix with no submatrix which is a row permutation of

$$q \cdot \mathbf{1}_2\mathbf{0}_2 = \overbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}}^{q}$$

Then

$$n \le \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \frac{q-3}{3}\binom{m}{2} + \binom{m}{m-2} + \binom{m}{m-1} + \binom{m}{m}$$

with equality only for

$$A = [K_m^0 K_m^1 K_m^2 T_{m,a} T_{m,b}^c K_m^{m-2} K_m^{m-1} K_m^m]$$

(for some choice a, b with $a + b = q - 3$)
if $m \ge q$ and $m \equiv 1, 3 (\mathrm{mod}\ 6)$.

Problem Let q be given. Does there exists an M so that for $m > M$, if A is an $m \times n$ simple matrix with no $4 \times q$ submatrix which is a row permutation of

$$q \cdot \mathbf{1}_3\mathbf{0}_1 = \overbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}}^{q}$$

Then

$$n \le \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \binom{m}{3} + \frac{q-3}{4}\binom{m}{3} + \binom{m}{m}$$

with equality only if there exists a simple $3 - (m, 4, \lambda)$ design with $\lambda = q - 2$?

Theorem (Keevash 14) Let $1/m \ll \theta \ll 1/k \leq 1/(t+1)$ and $\theta \ll 1$. Suppose that $\binom{k-i}{t-i}$ divides $\binom{m-i}{t-i}$ for $0 \leq i \leq r-1$. Then there exists a t-(m, k, λ) simple design for $\lambda \leq \theta m^{k-t}$.

Our Extremal Problem

Definition We say that a matrix A is *simple* if it is a (0,1)-matrix with no repeated columns.

Definition We define $\|A\|$ to be the number of columns in A.

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$\text{Avoid}(m, F) = \{\, A : A \text{ is } m\text{-rowed simple}, F \nprec A\}$

$\text{forb}(m, F) = \max_A\{\|A\| : A \in \text{Avoid}(m, F)\}$

Nearly exact bounds

$$F = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Forbidding F forces that the columns of any $A \in \text{Avoid}(m, F)$ have the property of being 2-laminar when viewed as sets.

Theorem (Dukes 14)

$$1.3818 \leq \limsup_{m\to\infty} \frac{\text{forb}(m, F)}{\binom{m}{2}} \leq 1.3821$$

Asymptotic Bounds

We are interested in $\text{forb}(m, s \cdot F)$. An example:

$$\text{Let} \quad F = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $\text{forb}(m, F)$ is $O(m^2)$. Now $s \cdot \mathbf{1}_3 \prec s \cdot F$ and so $\text{forb}(m, s \cdot F) \geq \text{forb}(m, s \cdot \mathbf{1}_3)$ (for any s).

Theorem Let $\alpha > 0$ be given. Then $\text{forb}(m, m^\alpha \cdot F)$ is $\Theta(m^{3+\alpha})$.

The upper bound is a challenge but the lower bound corresponds to constructing an $A \in \text{Avoid}(m, m^\alpha \cdot \mathbf{1}_3)$ with $\|A\|$ being $\Omega(m^{3+\alpha})$.

$s \cdot \begin{bmatrix}1\\1\end{bmatrix}$

We find that $[K_m^0 K_m^1 K_m^2 K_m^3] \in \text{Avoid}(m, m \cdot \begin{bmatrix}1\\1\end{bmatrix})$ and then we can show (by pigeonhole principle) that:

Theorem $\text{forb}(m, m \cdot \begin{bmatrix}1\\1\end{bmatrix}) = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \binom{m}{3}$.

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We find that $[K_m^0 K_m^1 K_m^2 K_m^3 K_m^4] \in \text{Avoid}(m, (m + \binom{m-2}{2})) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix})$ and then we can show (by pigeonhole principle) that:

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Thus $\text{forb}(m, (m + \binom{m-2}{2})) \cdot \begin{bmatrix}1\\1\end{bmatrix})$ is $\Theta(m^4)$.

Can we deduce the growth of $\text{forb}(m, m^\alpha \cdot \begin{bmatrix}1\\1\end{bmatrix})$?

Simple Triple Systems

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Let $T_{m,\lambda}$ denote the element-triple incidence matrix of a simple $2-(m, 3, \lambda)$ design. Thus $T_{m,\lambda}$ is an $m \times \frac{\lambda}{3}\binom{m}{2}$ simple matrix with all columns of column sum 3 and $T_{m,\lambda} \in \text{Avoid}(m, (\lambda+1) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix})$

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Let $T_{m,\lambda}$ denote the element-triple incidence matrix of a simple $2 - (m, 3, \lambda)$ design. Thus $T_{m,\lambda}$ is an $m \times \frac{\lambda}{3}\binom{m}{2}$ simple matrix with all columns of column sum 3 and $T_{m,\lambda} \in \text{Avoid}(m, (\lambda + 1) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix})$

Thus, choosing $\lambda = m^{1/2} - 2$, we have
$\text{forb}(m, m^{1/2} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix})$ is $\Theta(m^{5/2})$
or more generally, $\text{forb}(m, m^{\alpha} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix})$ is $\Theta(m^{2+\alpha})$ for $0 < \alpha \leq 1$.

Theorem (Keevash 14) Let $1/m \ll \theta \ll 1/k \leq 1/(t+1)$ and $\theta \ll 1$. Suppose that $\binom{k-i}{t-i}$ divides $\binom{m-i}{t-i}$ for $0 \leq i \leq r-1$. Then there exists a t-(m,k,λ) simple design for $\lambda \leq \theta m^{k-t}$.

This covers a fraction θ of the possible range for $\lambda \in \left(0, \binom{m}{k}\binom{k}{t}/\binom{m}{t}\right)$.

Let $\mathbf{1}_t$ denote the column of t 1's. The following result follows from Keevash 14.

Weak Packing: Let α and t be given. There exist a constant $c_{\alpha,t} > 0$ so that

$$\text{forb}(m, m^{\alpha} \cdot \mathbf{1}_t) \geq c_{\alpha,t} m^{t+\alpha}$$

i.e. $\text{forb}(m, m^{\alpha} \cdot \mathbf{1}_t)$ is $\Theta(m^{t+\alpha})$

We form a matrix in $\text{Avoid}(m, m^{\alpha} \cdot \mathbf{1}_t)$ by first taking all columns up to some appropriate size k, and then use the Weak Packing of $k+1$-sets that follows as a Corollary to Keevash' design result.

There are cases which do not yield the desired results.

$$\text{Let} \quad F = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Theorem (Frankl, Füredi, Pach 87) $\text{forb}(m, F) = \binom{m}{2} + 2m - 1$ i.e. $\text{forb}(m, F)$ is $O(m^2)$.

Theorem (A. and Lu 13) Let s be given. Then $\text{forb}(m, s \cdot F)$ is $\Theta(m^2)$.

Conjecture $\text{forb}(m, m^\alpha \cdot F)$ is $\Theta(m^{2+\alpha})$.

We can only prove that $\text{forb}(m, m^\alpha \cdot F)$ is $O(m^{3+\alpha})$.

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