Congaree

Being taught birdwatching by Jerry

Jerry isn't tall

Jerry and Jeannine in Magnolia Gardens

Deynise, Malia and Jerry

Forbidden Configurations

Richard Anstee,
UBC, Vancouver

University of South Carolina, April 27, 2018

Introduction

The paper ‘Small Forbidden Configurations’, joint with Jerry Griggs and Attila Sali, began a systematic exploration of the subject. The collaboration is from a sabbatical visit of Jerry to Vancouver and a visit of Attila in 1993. That paper contains the origin of the conjecture that I will describe.

Survey at www.math.ubc.ca/∼anstee

Simple Matrices and Set Systems

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i.e. if A is m-rowed then A is the incidence matrix of some family $\mathcal{A}$ of subsets of $[m] = \{1, 2, \ldots, m\}$.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathcal{A} = \big\{\emptyset, \{2\}, \{3\}, \{1, 3\}, \{1, 2, 3\}\big\}$$

Configurations

Definition Given a matrix F, we say that A has F as a *configuration* written $F \prec A$ if there is a submatrix of A which is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \prec \quad \left[\begin{array}{c|ccc|c|c|} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \color{red}{1} & \color{red}{0} & \color{red}{1} & 1 & \color{red}{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \color{red}{1} & \color{red}{1} & \color{red}{0} & 0 & \color{red}{0} \end{array}\right] = A$$

Our Extremal Problem

Definition We define $\|A\|$ to be the number of columns in A.

Let $\mathcal{F}$ be a family of (0,1)-matrices.

$\mathrm{Avoid}(m, \mathcal{F}) = \{\, A : A$ is m-rowed simple, $F \nprec A$ for $F \in \mathcal{F}\}$

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$\mathrm{forb}(m,\mathcal{F}) = \max_A\{\|A\| : A \in \mathrm{Avoid}(m,\mathcal{F})\}$

There are other possibilities for extremal problems for $\mathrm{Avoid}(m,\mathcal{F})$ including maximizing the weighted sum over columns where a column of column sum i is weighted by $1/\binom{m}{i}$ (e.g. Johnston and Lu) or maximizing the number of 1's .

A Product Construction

As with any extremal problem, the results are often motivated by constructions, namely matrices in $\mathrm{Avoid}(m, F)$. The early investigations with Jerry Griggs and Attila Sali suggested a product construction might be very helpful.

The building blocks of our product constructions are I, I^c and T:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A Product Construction

Definition Given an $m_1 \times n_1$ matrix A and a $m_2 \times n_2$ matrix B we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of A on top of a column of B. If A, B are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

$$\underset{A}{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}} \times \underset{B}{\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}} = \left[\begin{array}{ccc|ccc|ccc}1&1&1&0&0&0&0&0&0\\0&0&0&1&1&1&0&0&0\\0&0&0&0&0&0&1&1&1\\\hline 1&1&1&1&1&1&1&1&1\\0&1&1&0&1&1&0&1&1\\0&0&1&0&0&1&0&0&1\end{array}\right]$$

Given p simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the p-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. with $\Theta(m^p)$ columns.

The Conjecture

Definition Let $x(F)$ denote the largest p such that there is a p-fold product which does not contain F as a configuration where the p-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

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Conjecture (A, Sali 05) $\mathrm{forb}(m, F)$ *is* $\Theta(m^{x(F)})$.

In other words, we predict our product constructions with the three building blocks $\{I, I^c, T\}$ determine the asymptotically best constructions. The conjecture has now been verified in many cases.

Attila Sali

Exact bounds and asymptotic bounds

Definition Let $s \cdot F = [\overbrace{FF \cdots F}^{s}]$.

$$\text{Let} \quad F = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Theorem (Frankl, Füredi, Pach 87) $\mathrm{forb}(m, F) = \binom{m}{2} + 2m - 1$
i.e. $\mathrm{forb}(m, F)$ is $\Theta(m^2)$.

Theorem (A. and Lu 13) Let s be given. Then $\mathrm{forb}(m, s \cdot F)$ is $\Theta(m^2)$.

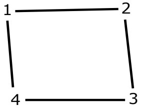
Note for this F, $x(F) = 2 = x(s \cdot F)$ for any constant s, so the result is evidence for the conjecture

Berge Hypergraphs

Claude Berge, and others, created hypergraphs as a generalization of graphs. There are several hypergraph generalizations of paths and cycles. One generalization yields Berge paths and cycles. The definition of Berge Hypergraphs was given to me by Gerbner and Palmer (2015) and follows the same ideas. With Santiago Salazar, we consider the extremal set problem obtained by forbidding a single Berge Hypergraph

Santiago Salazar

Let F be a hypergraph with edges $E_1, E_2, \ldots, E_\ell$. We say that a hypergraph H has F as a Berge Hypergraph and write $F \lll H$ if there are ℓ edges $E'_1, E'_2, \ldots, E'_\ell$ of H so that $E_i \subseteq E'_i$ for $i = 1, 2, \ldots, \ell$.



$$F = C_4$$
$$E_1 = \{1, 2\}$$
$$E_2 = \{2, 3\}$$
$$E_3 = \{3, 4\}$$
$$E_4 = \{1, 4\}$$

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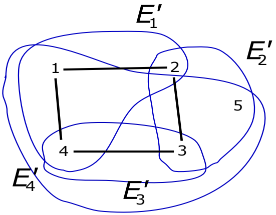

$F = C_4$	$\lll$	H
$E_1 = \{1, 2\}$		$E'_1 = \{1, 2, 4\}$
$E_2 = \{2, 3\}$		$E'_2 = \{2, 3, 5\}$
$E_3 = \{3, 4\}$		$E'_3 = \{3, 4\}$
$E_4 = \{1, 4\}$		$E'_4 = \{1, 3, 4, 5\}$

Berge Hypergraphs

$$C_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \ll \begin{array}{c} \begin{array}{cccccc} E_1' & E_2' & E_3' & E_4' & & \end{array} \\ \begin{bmatrix} \color{red}{1} & 0 & 0 & \color{red}{1} & \cdots \\ \color{red}{1} & \color{red}{1} & 0 & 0 & \cdots \\ 0 & \color{red}{1} & \color{red}{1} & \color{blue}{1} & \cdots \\ \color{blue}{1} & 0 & \color{red}{1} & \color{red}{1} & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \end{array}$$

1's matter in C_4 when considering a Berge hypergraph of C_4, but 0's in C_4 don't matter.

Berge Hypergraphs

Define our extremal problem as follows:

$$\mathrm{BergeAvoid}(m, F) = \{A \,:\, A \text{ is } m\text{-rowed, simple, } F \not\prec A\},$$

$$\mathrm{Bforb}(m, F) = \max_{A}\{\|A\| \,:\, A \in \mathrm{BergeAvoid}(m, F)\}.$$

Downsets

Theorem If $A \in \mathrm{BergeAvoid}(m, F)$, then there exists an $A' \in \mathrm{BergeAvoid}(m, F)$ with $\|A\| = \|A'\|$ and the columns of A' form a downset: namely if α is a column of A' and $\beta \leq \alpha$, then β is also a column of A'.

Proof: Apply a shifting argument, replacing 1's by 0's in A as long as no repeated columns are created. The result is A'.

Theorem $\mathrm{Bforb}(m, I_k) = 2^{k-1}$

Theorem $\mathrm{Bforb}(m, C_4) = \Theta(m^{3/2})$

Note that $I_2 \times I_2 \approx C_4 \approx K_{2,2}$

Theorem Let $t \geq 3$. Then $\mathrm{Bforb}(m, I_3 \times I_t) = \Theta(m^2)$

For this latter result we needed recent extremal graph results. Note that $I_3 \times I_t$ is the vertex-edge incidence matrix of $K_{3,t}$.

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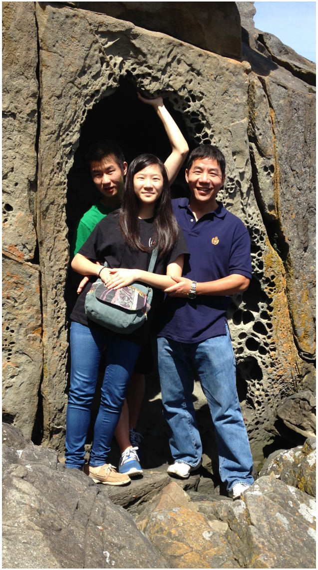
Definition $\mathrm{ex}(m, K_\ell, K_{s,t})$ is the maximum number of copies of K_ℓ in an m-vertex $K_{s,t}$-free graph.

Such an extremal function has been studied, with surprisingly good results obtained, by Alon and Shikhelman '15 and Kostachka, Mubayi and Verstratte '15.

Theorem (Alon, Shikhelman '15, Kostochka, et al '15)
Let s, t be given with $t \geq (s-1)! + 1$. Then
$\mathrm{ex}(m, K_3, K_{s,t})$ is $\Theta(m^{3-(3/s)})$.



Linyuan and his kids on Pender Island

An Unavoidable Forbidden Family

Theorem (Balogh and Bollobás 05) Let k be given. Then

$$\mathrm{forb}(m, \{I_k, I_k^c, T_k\}) \leq 2^{2^k}$$

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If you take all columns of column sum at most $k-1$ that arise from the $k-1$-fold product $T_{k-1} \times T_{k-1} \times \cdots \times T_{k-1}$ then this yields $\binom{2k-2}{k-1} \approx 2^{2k}$ columns. A probabalistic construction in $\mathrm{Avoid}(m, \{I_k, I_k^c, T_k\})$ has $2^{ck \log k}$ columns.

Ramsey Theory

Proofs used lots of induction and multicoloured Ramsey numbers: $R(k_1, k_2, \ldots, k_\ell)$ is the smallest value of n such than any colouring of the edges of K_n with ℓ colours $1, 2, \ldots, \ell$ will have some colour i and a clique of k_i vertices with all edges of colour i. These numbers are readily bounded by multinomial coefficients:

$$R(k_1, k_2, \ldots, k_\ell) \leq \binom{\sum_{i=1}^{\ell} k_i}{k_1\ k_2\ k_3 \cdots\ k_\ell}$$

$$R(k_1, k_2, \ldots, k_\ell) \leq \ell^{k_1+k_2+\cdots+k_\ell}$$

Our first proof had something like $\mathrm{forb}(m\{, I_k, I_k^c, T_k\}) < R(R(k,k), R(k,k))$ yielding a doubly exponential bound.

We say a matrix with entries in $\{0, 1, \ldots, r-1\}$ is an r-matrix.

An r-matrix is simple if there are no repeated columns.

$\text{forb}(m, r, \mathcal{F}) = \max\{\|A\| : A \text{ is simple } r\text{-matrix}, F \not\prec A \;\; \forall F \in \mathcal{F}\}$

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$$\text{Let } T_k(a,b,c) = \left.\begin{bmatrix} b & c & c & \cdots & c \\ a & b & c & \cdots & c \\ a & a & b & \cdots & c \\ \vdots & \vdots & \vdots & \ddots & \\ a & a & a & \cdots & b \end{bmatrix}\right\} k$$

$$\text{Let } \mathcal{T}_k(r) = \{T_k(a,b,a) : a \neq b, \quad a, b \in \{0, 1, \ldots, r-1\}\}$$

$$\cup \{T_k(a,b,b) : a \neq b, \quad a, b \in \{0, 1, \ldots, r-1\}\}$$

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Theorem (A, Lu 14) Given r there exists a constant c_r so that $\mathrm{forb}(m, r, \mathcal{T}_k(r)) \leq 2^{c_r k^2}$.

Using Ramsey Theory

Consider 3-matrices, that is matrices with entries in $\{0,1,2\}$. By Ramsey Theory, if $n \geq R(k,k,k)$, then any choices for the entries marked $*$ in the $n \times n$ matrix

$$\left.\begin{bmatrix} b & * & * & \cdots & * \\ a & b & * & \cdots & * \\ a & a & b & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \\ a & a & a & \cdots & b \end{bmatrix}\right\} n$$

we will find one of the configurations $T_k(a,b,0)$ or $T_k(a,b,1)$ or $T_k(a,b,2)$.

$$\mathcal{T}_k(2) =$$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 0 \end{bmatrix}.$$

$$\mathcal{T}_k(2) \approx \{I_k, I_k^c, T_k\}$$

$\mathcal{T}_k(3)\backslash\mathcal{T}_k(2) =$

$$\begin{bmatrix} 1 & 2 & \cdots & 2 \\ 2 & 1 & \cdots & 2 \\ \vdots & \vdots & \ddots & \\ 2 & 2 & \cdots & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 & \cdots & 2 \\ 2 & 0 & \cdots & 2 \\ \vdots & \vdots & \ddots & \\ 2 & 2 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 2 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 & \cdots & 2 \\ 0 & 2 & \cdots & 2 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 & \cdots & 2 \\ 1 & 2 & \cdots & 2 \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 2 \end{bmatrix},$$

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Jeffrey Dawson

Awkward extra matrix

$$T_k(0,2,1) = \begin{bmatrix} 2 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 1 & \cdots & 1 \\ 0 & 0 & 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix}$$

Theorem Let $\mathcal{F}$ be a family of $(0,1)$-matrices. $\mathrm{forb}(m, 3, (\mathcal{T}_k(3)\backslash\mathcal{T}_k(2) \cup T_k(0,2,1) \cup \mathcal{F}))$ is $\Theta(\mathrm{forb}(m, \mathcal{F}))$.

Surely $T_k(0,2,1)$ is not needed for this result. Dawson, Lu, Sali and A. '17 have some preliminary results on eliminating $T_k(0,2,1)$. Our results have made heavy use of Ramsey Theory.

Eliminating $T_k(0,2,1)$

Corollary Let $F \prec T_k(0,2,1)$. Then $\mathrm{forb}(m,3,(\mathcal{T}_k(3)\backslash\mathcal{T}_k(2)\cup F))$ is $\Theta(\mathrm{forb}(m,F))$.

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Corollary $\mathrm{forb}(m,3,(\mathcal{T}_k(3)\backslash\mathcal{T}_k(2)\cup[0\ 1]))$ is $\Theta(1)$

Eliminating $T_k(0, 2, 1)$

Corollary Let $F \prec T_k(0, 2, 1)$. Then $\text{forb}(m, 3, (\mathcal{T}_k(3)\backslash\mathcal{T}_k(2) \cup F))$ is $\Theta(\text{forb}(m, F))$.

Corollary $\text{forb}(m, 3, (\mathcal{T}_k(3)\backslash\mathcal{T}_k(2) \cup [0\ 1]))$ is $\Theta(1)$

Theorem $\text{forb}(m, 3, (\mathcal{T}_k(3)\backslash\mathcal{T}_k(2) \cup \emptyset))$ is $\Theta(2^m)$ which is $\Theta(\text{forb}(m, \emptyset))$.

Eliminating $T_k(0,2,1)$

Corollary Let $F \prec T_k(0,2,1)$. Then $\mathrm{forb}(m,3,(\mathcal{T}_k(3)\backslash\mathcal{T}_k(2)\cup F))$ is $\Theta(\mathrm{forb}(m,F))$.

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Theorem $\mathrm{forb}(m,3,(\mathcal{T}_k(3)\backslash\mathcal{T}_k(2)\cup\emptyset))$ is $\Theta(2^m)$ which is $\Theta(\mathrm{forb}(m,\emptyset))$.

Theorem $\mathrm{forb}(m,3,(\mathcal{T}_k(3)\backslash\mathcal{T}_k(2)\cup I_2))$ is $\Theta(\mathrm{forb}(m,I_2))$.

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Corollary Let $F \prec T_k(0,2,1)$. Then $\mathrm{forb}(m,3,(\mathcal{T}_k(3)\backslash\mathcal{T}_k(2)\cup F))$ is $\Theta(\mathrm{forb}(m,F))$.

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A nice inductive result:

Theorem $\mathrm{forb}(m,3,(\mathcal{T}_k(3)\backslash\mathcal{T}_k(2)\cup \begin{bmatrix}1\\0\end{bmatrix} \times F))$ is $\Theta(m\cdot\mathrm{forb}(m,3,(\mathcal{T}_k(3)\backslash\mathcal{T}_k(2)\cup F))$.

Congratulations on this milestone.
And thank you, Jerry, for your friendship over the years.