Two Extremal Set Results

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Introduction

We begin with some helpful notations.

Definition $[m] = \{1, 2, \ldots, m\}$

Definition $2^{[m]} = \{A \,|\, A \subseteq [m]\}$ or power set of $[m]$

Definition $A^c = [m] \backslash A$ or complement of A

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'$\subset$' means contained in, '$\emptyset$' is the empty set, '$A \cap B$' is intersection of A, B

Theorem Let $\mathcal{F} \subseteq 2^{[m]}$. Assume for all pairs $A, B \in \mathcal{F}$, we have $A \cap B \neq \emptyset$. Then

$$|\mathcal{F}| \leq 2^{m-1}.$$

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Proof We can partition $2^{[m]}$ into 2^{m-1} pairs of sets A, A^c. At most one of the two sets A, A^c can be in $\mathcal{F}$ since $A \cap A^c = \emptyset$. Thus at most half the sets in $2^{[m]}$ can be in $\mathcal{F}$, proving the bound. ■

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I was at a talk where Peter Frankl called this Theorem 0 of Extremal Set Theory. Peter Frankl is perhaps the world's most famous (living) Mathematician since he is a media personality in Japan

Sperner's Theorem

Definition Let $\mathcal{F} \subseteq 2^{[m]}$. We say $\mathcal{F}$ is an antichain if for any pair $A, B \in \mathcal{F}$ neither $A \subset B$ nor $B \subset A$.

Theorem (Sperner 1927) Let $\mathcal{F} \subseteq 2^{[m]}$ and assume $\mathcal{F}$ is an antichain. Then

$$|\mathcal{F}| \leq \binom{m}{\lfloor m/2 \rfloor}.$$

Emanuel Sperner

1905 – 1980

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We can achieve the bound by taking all subsets of $[m]$ of size $\lfloor m/2 \rfloor$.
Note $\lfloor m/2 \rfloor$ is the greatest integer at most $m/2$, sometimes called the floor of $m/2$.

Definition A chain is a sequence $A_1 \subset A_2 \subset \cdots \subset A_k$ of subsets of $[m]$.
Definition We say a chain is saturated if $|A_{i+1}| = |A_i| + 1$ for $i = 1, 2, \ldots, k-1$.
Definition We say a chain is symmetric if $|A_i| = m - |A_{k-i+1}|$ i.e. symmetric about $\lfloor m/2 \rfloor$.

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Proof of Sperner's Theorem. We wish to partition $2^{[m]}$ into $\binom{m}{\lfloor m/2 \rfloor}$ saturated symmetric chains. Two elements of an antichain cannot be together in any chain; at most one element of $\mathcal{F}$ can come from a chain. The chains are saturated and symmetric and hence have at least one set of size $\lfloor m/2 \rfloor$. This yields the bound if we could find the partition.

We now seek the partition.

Proof continued

We use induction on m to obtain the partition. Assume we have the appropriate partition for $2^{[m]}$ with symmetric saturated chains $A_1 \subset A_2 \subset \cdots \subset A_k$ and we will obtain the appropriate partition for $2^{[m+1]}$.

We first make the observation that every set in $2^{[m+1]}$ either contains $m+1$ or does not and hence we can obtain $2^{[m+1]}$ from $2^{[m]}$ as follows. For each set $A \in 2^{[m]}$, we form two sets $A, A \cup \{m+1\}$.

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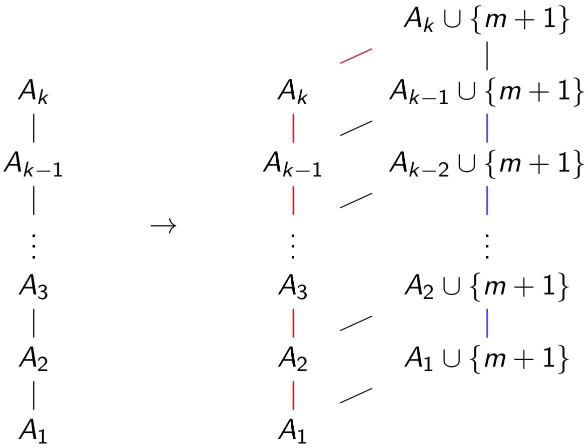
The chain $A_1 \subset A_2 \subset \cdots \subset A_k$ yields the $2k$ sets $A_1, A_2, \ldots, A_k$ and $A_1 \cup \{m+1\}, A_2 \cup \{m+1\}, \ldots, A_k \cup \{m+1\}$. We can readily partition these $2k$ sets into two chains, one of size $k+1$ and one of size $k-1$ as follows: First chain is $A_1 \subset A_2 \subset \cdots \subset A_k \subset A_k \cup \{m+1\}$ and second chain is $A_1 \cup \{m+1\} \subset A_2 \cup \{m+1\} \subset \cdots \subset A_{k-1} \cup \{m+1\}$ which we can verify are saturated chains and given that our original chain is symmetric, our new chain is symmetric with m replaced by $m+1$.



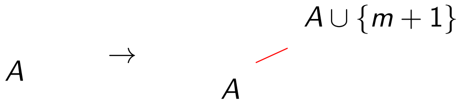
The red lines and the blue lines mark the two new (saturated, symmetric) chains obtained from the single chain $A_1 \subset A_2 \subset \cdots \subset A_k$ after adding element $m+1$.

It is possible, when m is even, that a symmetric saturated chain consists of a single set of size $m/2$. In fact simple counting shows that this must be true, namely some of the chains in the decomposition would consist of a single set. Say the chain consists ofthe single set B with $|B| = m/2$. Then, we have the two sets $B, B \cup \{m+1\}$ and $B \subset B \cup \{m+1\}$ forms a symmetric(!) saturated chain in $2^{[m+1]}$.

You will note that if we start with a chain of size 2, then it gives rise to two symmetric saturated chains in $2^{[m+1]}$, one of size 3 and one of size 1.

m is even. Some special cases.

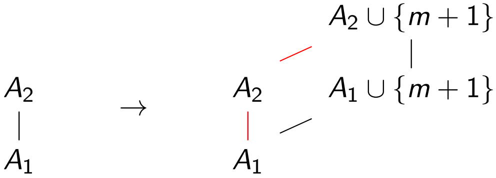
Thus $m/2$ is an integer. Say A is a set of size $m/2$ and is the sole element in a symmetric saturated chain of $2^{[m]}$. We can proceed as before



The red line marks the new saturated symmetric chain $(A \subset A \cup \{m+1\})$.

m is odd. Some special cases.

Thus $m/2$ is an integer. Say A_1, A_2 be sets of size $(m-1)/2, (m-1)/2+1$ respectively that form a symmetric saturated chain $(A_1 \subset A_2)$ of $2^{[m]}$ of 2 sets. We can proceed as before



The red lines mark the new (saturated, symmetric) chain of 3 sets $(A_1 \subset A_2 \subset A_2 \cup \{m+1\})$ and we also have the single chain consisting of the set $A_1 \cup \{m+1\}$.

Thanks for your attention