Forbidden Configurations
A shattered history

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I have had the good fortune of working with a number of coauthors in this area: Farzin Barekat, Jeffrey Dawson, Kim Dinh, Laura Dunwoody, Ron Ferguson, Balin Fleming, Zoltan Füredi, Jerry Griggs, Nima Kamoosi, Steven Karp, Peter Keevash, Christina Koch, Linyuan (Lincoln) Lu, Connor Meehan, U.S.R. Murty, Niko Nikov, Zachary Pellegrin, Miguel Raggi, Lajos Ronyai, Santiago Salazar, Attila Sali, Cindy Tan.

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i.e. if A is m-rowed then A is the incidence matrix of some family $\mathcal{A}$ of subsets of $[m] = \{1, 2, \ldots, m\}$.

$$A = \begin{bmatrix} 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathcal{A} = \{\emptyset, \{2\}, \{3\}, \{1, 3\}, \{1, 2, 3\}\}$$

Definition of a Configuration

Definition Given a matrix F, we say that A has F as a *configuration* written $F \prec A$ if there is a submatrix of A which is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \prec \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \quad 0 \quad 1} & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \quad 1 \quad 0} & & 0 & \boxed{0} \end{bmatrix} = A$$

Our Extremal Problem

Definition We define $\|A\|$ to be the number of columns in A.

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$$\text{forb}(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = m + 1.$$

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible columns on k rows.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} = \Theta(m^{k-1})$$

We say a set of rows S is shattered by A if $K_{|S|} \prec A|_S$.

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VC-dimension appears in many results but most remarkably (for me) in machine learning.

$$\text{Let } sh(A) = \{S \subseteq [m] : A \text{ shatters } S\}$$

e.g.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

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So $|sh(A)| = 7 \geq 6 = \|A\|$

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Theorem (Pajor 85) $\quad |sh(A)| \geq \|A\|$.

Proof: Decompose A as follows:

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$|sh(A)| \geq |sh(A_0)| + |sh(A_1)|$.
Hence $|sh(A)| \geq \|A\|$. ■

Remark If A shatters S then A shatters any subset of S.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$\text{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}$$

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Proof: Let $A \in \text{Avoid}(m, K_k)$.
Then $sh(A)$ can only contain sets of size $k-1$ or smaller.
Then

$$\binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \geq |sh(A)| \geq \|A\|.$$

Critical Substructures

Definition A *critical substructure* of a configuration F is a minimal configuration $F' \prec F$ such that

$$\text{forb}(m, F') = \text{forb}(m, F).$$

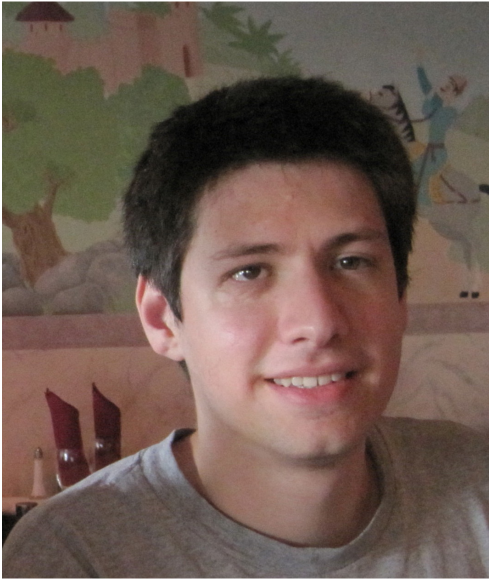
When $F' \prec F'' \prec F$, we deduce that

$$\text{forb}(m, F') = \text{forb}(m, F'') = \text{forb}(m, F).$$

Let $\mathbf{1}_k\mathbf{0}_\ell$ denote the $(k+\ell) \times 1$ column of k 1's on top of ℓ 0's.
Let K_k^ℓ denote the $k \times \binom{k}{\ell}$ simple matrix of all columns of sum ℓ.

Miguel Raggi

Steven Karp



Miguel Raggi

Steven Karp

Definition If A is $m \times n$, then $t \cdot A = [A\,A\,\cdots\,A]$ is $m \times tn$.

Critical Substructures for K_4

$$K_4 = \begin{bmatrix} 1&1&1&1&0&1&1&1&0&0&0&1&0&0&0&0\\ 1&1&1&0&1&1&0&0&1&1&0&0&1&0&0&0\\ 1&1&0&1&1&0&1&0&1&0&1&0&0&1&0&0\\ 1&0&1&1&1&0&0&1&0&1&1&0&0&0&1&0 \end{bmatrix}$$

Critical substructures are $\mathbf{1}_4$, K_4^3, K_4^2, K_4^1, $\mathbf{0}_4$, $2\cdot\mathbf{1}_3$, $2\cdot\mathbf{0}_3$.
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The same is conjectured to be true for K_k for $k \geq 5$.

We can extend K_4 and yet have the same bound

$[K_4|\mathbf{1}_2\mathbf{0}_2] =$

$$\left[\begin{array}{cccccccccccccccc|c} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

Theorem (A., Meehan 11) For $m \geq 5$, we have $\text{forb}(m, [K_4|\mathbf{1}_2\mathbf{0}_2]) = \text{forb}(m, K_4)$.

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$\text{forb}(m, [K_4|\mathbf{1}_2\mathbf{0}_2]) = \text{forb}(m, K_4)$.
We expected in fact that we could add many copies of the column $\mathbf{1}_2\mathbf{0}_2$ and obtain the same bound, albeit for larger values of m.

Connor Meehan

Connor Meehan

We can extend K_4 further and yet have the same bound

$[K_4|t\cdot K_2^T] =$

$$\left[\begin{array}{cccccccccccccccc|c}
1&1&1&1&0&1&1&1&0&0&0&1&0&0&0&0& \\
1&1&1&0&1&1&0&0&1&1&0&0&1&0&0&0& \\
1&1&0&1&1&0&1&0&1&0&1&0&0&1&0&0& t\cdot\begin{bmatrix}1&1\\1&0\\0&1\\0&0\end{bmatrix} \\
1&0&1&1&1&0&0&1&0&1&1&0&0&0&1&0&
\end{array}\right]$$

Theorem (A., Nikov 21) There exits a constant N_t so that for $m \geq N_t$, then $\text{forb}(m,[K_4|t\cdot K_2^T]) = \text{forb}(m,K_4)$.

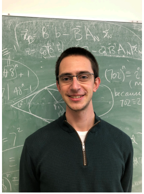
We can extend K_4 further and yet have the same bound

$[K_4 | t \cdot K_2^T] =$

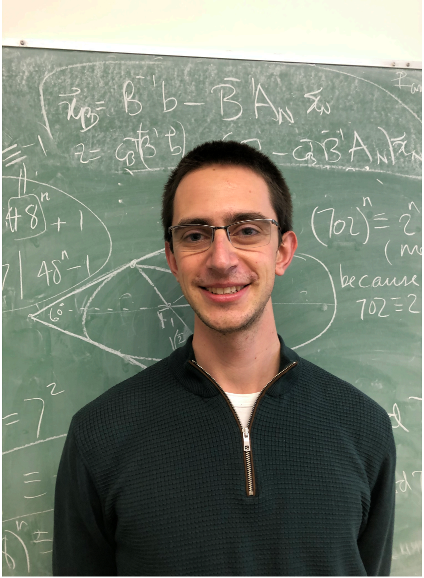
$$\left[\begin{array}{cccccccccccccccc|c} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & t \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & \end{array}\right]$$

Theorem (A., Nikov 21) There exits a constant N_t so that for $m \geq N_t$, then $\text{forb}(m, [K_4 | t \cdot K_2^T]) = \text{forb}(m, K_4)$.

It is possible that as many as 5 different columns, each with 2 1's, can be added to K_4 but adding K_4^2 increases bound to $\Theta(m^4)$.



Niko Nikov



Niko Nikov

Exact Bounds

Theorem (A., Füredi 84) $\text{forb}(m, \mathbf{1}_k) = \text{forb}(m, K_k)$ and $\text{forb}(m, t \cdot \mathbf{1}_k) = \text{forb}(m, t \cdot K_k)$. ■

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Theorem (A, Barekat, Pellegrin 19) Let k, ℓ, t be given with $k > \ell$. Then for m large,
$\text{forb}(m, t \cdot \mathbf{1}_k\mathbf{0}_\ell) = \text{forb}(m, t \cdot K_k) + \sum_{i=m-\ell+1}^{m} \binom{m}{i}$. ■

Note that for small m, the bounds do not hold. The gap was small and we could use the existence of certain structures when we were close to the bound.

Zachary Pellegrin

Zachary Pellegrin

Further extensions to K_k, Asymptotic Bounds

With Attila Sali, we published a conjecture in 2005 about what properties drive the asympotics of forb(m, F). Our conjecture says that you only have to look at a small number of possible constructions as candidates in Avoid(m, F). Students have made many contributions. It is still a conjecture!

Further extensions to K_k, Asymptotic Bounds

Let B be a $k \times (k+1)$ matrix which has one column of each column sum. Given two matrices C, D, let $C \backslash D$ denote the matrix obtained from C by deleting any columns of D that are in C (i.e. set difference). Let

$F_B(t) = [K_k | t \cdot [K_k \backslash B]]$.

Theorem (A, Griggs, Sali 97, A, Sali 05,
A, Fleming, Füredi, Sali 05)
forb$(m, F_B(t))$ *and* *forb*(m, K_k) *are both* $\Theta(m^{k-1})$.

The difficult problem here was the bound with either linear algebra or induction proofs.

Let D be the $k \times (2^k - 2^{k-2} - 1)$ simple matrix with all columns of sum at least 1 that do not simultaneously have 1's in rows 1 and 2. We take $F_D(t) = [\mathbf{0}_k \; (t+1)\cdot D]$ which for $k = 4$ becomes

$$F_D(t) = \left[\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \; (t+1)\cdot \left[\begin{matrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{matrix} \right] \right]$$

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Theorem (A, Sali 05 (for $k = 3$), A, Fleming 09)
forb$(m, F_D(t))$ *is* $\Theta(m^{k-1})$.

The argument used standard results for directed graphs, *indicator polynomials* and a linear algebra rank argument

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The argument used standard results for directed graphs, *indicator polynomials* and a linear algebra rank argument

Theorem *Let k be given and assume F is a k-rowed configuration which is not a configuration in $F_B(t)$ for any choice of B as a $k \times (k+1)$ simple matrix with one column of each column sum and not in $F_D(t)$, for any t. Then forb(m, F) is $\Theta(m^k)$.*

Asymptotic Bounds

$$F_{10} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Theorem (A., Sali, Tan, White 18) forb(m, F_{10}) is $\Theta(m^2)$. ■

We generalized a previous proof for another 5×6 forbidden configuration that also resulted in a $\Theta(m^2)$ bound.

CindyTan

CindyTan

More Questions

K_3^T is the 8×3 transpose of K_3.

Theorem (Keevash et al 19) forb(m, K_3^T) is $\Theta(m^3)$.

How does this fit in with the conjecture?

Kim Dinh

Kim Dinh

The following matrices are important:

$$G_{6\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad I_2 \times G_{6\times 3} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Theorem (A., Raggi, Sali) $\text{forb}(m, G_{6\times 3})$ is $\Theta(m^2)$. ■

Theorem (A., Dinh 20) Our conjecture predicts that $\text{forb}(m, I_2 \times G_{6\times 3})$ is $\Theta(m^3)$ and any 8-rowed F with $\text{forb}(m, F)$ being $O(m^3)$ must have $F \prec I_2 \times G_{6\times 3}$. Adding any column α to $I_2 \times G_{6\times 3}$ results in $\text{forb}(m, [\alpha\ I_2 \times G_{6\times 3}])$ being $\Omega(m^4)$.

The following matrices are important:

$$G_{6\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad I_2 \times G_{6\times 3} = \left[\begin{array}{c|ccc|cc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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Note that $K_3^T \prec I_2 \times G_{6\times 3}$ in columns 2,3,4 of $I_2 \times G_{6\times 3}$

There is lots more work to be done