Forbidden Families of Configurations

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Consider the following family of subsets of $\{1, 2, 3, 4\}$:

$$\mathcal{A} = \{\emptyset, \{1, 2, 4\}, \{1, 4\}, \{1, 2\}, \{1, 2, 3\}, \{1, 3\}\}$$

The incidence matrix A of the family $\mathcal{A}$ of subsets of $\{1, 2, 3, 4\}$ is:

$$A = \begin{bmatrix} 0 & 1 & \boxed{1} & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Definition We say that a matrix A is *simple* if it is a (0,1)-matrix with no repeated columns.

Definition We define $\|A\|$ to be the number of columns in A.

$$\|A\| = 6 = |\mathcal{A}|$$

Definition Given a matrix F, we say that A has F as a *configuration* (denoted $F \prec A$) if there is a submatrix of A which is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \prec \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \quad 0 \quad 1} & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \quad 1 \quad 0} & & 0 & \boxed{0} \end{bmatrix}$$

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Definitions

$\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$

$\text{Avoid}(m, \mathcal{F}) = \{\, A \,:\, A\ m\text{-rowed simple},\ F \not\prec A \text{ for all } F \in \mathcal{F}\}$

$\mathit{forb}(m, \mathcal{F}) = \max_A\{\|A\| : A \in \text{Avoid}(m, \mathcal{F})\}$

Main Bounds

Definition Let K_k be the $k \times 2^k$ simple matrix of all possible columns on k rows.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$

Theorem (Füredi 83). *Let F be a $k \times \ell$ matrix. Then $forb(m, F) = O(m^k)$.*

Problem Given F, can we predict the behaviour of $forb(m, F)$?

Let C_k denote the $k \times k$ vertex-edge incidence matrix of the cycle of length k.

$$\text{e.g.}\quad C_3 = \begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}, C_4 = \begin{bmatrix}1 & 0 & 0 & 1\\ 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}.$$

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Matrices in $\text{Avoid}(m, \{C_3, C_5, C_7, \ldots\})$ are called
Balanced Matrices.
Theorem $\mathit{forb}(m, \{C_3, C_5, C_7, \ldots\}) = \mathit{forb}(m, C_3)$

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Matrices in Avoid$(m, \{C_3, C_4, C_5, C_6, \ldots\})$ are called
Totally Balanced Matrices.
Theorem $\mathit{forb}(m, \{C_3, C_4, C_5, C_6, \ldots\}) = \mathit{forb}(m, C_3)$

The inequality $\mathit{forb}(m, \{C_3, C_4, C_5, C_6, \ldots\}) \leq \mathit{forb}(m, C_3)$ follows from the following:

Lemma If $\mathcal{F}' \subset \mathcal{F}$ then $\mathit{forb}(m, \mathcal{F}) \leq \mathit{forb}(m, \mathcal{F}')$.

The equality follows from a result that any $m \times \mathit{forb}(m, C_3)$ simple matrix is in fact totally balanced (A, 80).
Thus we conclude
$\mathit{forb}(m, \{C_3, C_4, C_5, C_6, \ldots\}) = \mathit{forb}(m, C_3)$.

A Product Construction

The building blocks of our product constructions are I, I^c and T:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Definition Given an $m_1 \times n_1$ matrix A and a $m_2 \times n_2$ matrix B we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of A on top of a column of B. If A, B are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \times \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix} = \begin{bmatrix}1&1&1&0&0&0&0&0&0\\0&0&0&1&1&1&0&0&0\\0&0&0&0&0&0&1&1&1\\1&1&1&1&1&1&1&1&1\\0&1&1&0&1&1&0&1&1\\0&0&1&0&0&1&0&0&1\end{bmatrix}$$

Given p simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the p-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.

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The Conjecture

Definition Let $x(F)$ denote the smallest p such that every p-fold product contains F as a configuration where the p-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

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In other words, we predict our product constructions with the three building blocks $\{I, I^c, T\}$ determine the asymptotically best constructions.

The conjecture has been verified for $k \times \ell$ F where $k = 2$ (A, Griggs, Sali 97) and $k = 3$ (A, Sali 05) and $\ell = 2$ (A, Keevash 06) and for k-rowed F with bounds $\Theta(m^{k-1})$ or $\Theta(m^k)$ plus other cases.

Forbidden Families can fail Conjecture

Definition $\text{ex}(m, H)$ is the maximum number of edges in a (simple) graph G on m vertices that has no subgraph H.

$A \in \text{Avoid}(m, \mathbf{1}_3)$ will be a matrix with up to $m+1$ columns of sum 0 or sum 1 plus columns of sum 2 which can be viewed as the vertex-edge incidence matrix of a graph.
Let $I(H)$ denote the $|V(H)| \times |E(H)|$ vertex-edge incidence matrix associated with H.
Theorem $\mathit{forb}(m, \{\mathbf{1}_3, I(H)\}) = m + 1 + \text{ex}(m, H)$.

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Theorem $\mathit{forb}(m, \{\mathbf{1}_3, I(H)\}) = m + 1 + \text{ex}(m, H)$.
In this talk $I(C_4) = C_4$, $I(C_6) = C_6$.
Theorem $\mathit{forb}(m, \{\mathbf{1}_3, C_4\}) = m + 1 + \text{ex}(m, C_4)$ which is $\Theta(m^{3/2})$.

Theorem $\mathit{forb}(m, \{\mathbf{1}_3, C_6\}) = m + 1 + \text{ex}(m, C_6)$ which is $\Theta(m^{4/3})$.

Forbidden Families can pass Conjecture

Theorem (Balogh and Bollobás 05) Let k be given. Then there is a constant c_k so that $forb(m, \{I_k, I_k^c, T_k\}) = c_k$.

We note that there is no obvious product construction.

Note that $c_k \geq \binom{2k-2}{k-1}$ by taking all columns of column sum at most $k-1$ that arise from the $k-1$-fold product $T_{k-1} \times T_{k-1} \times \cdots \times T_{k-1}$.

Let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ and $\mathcal{G} = \{G_1, G_2, \ldots, G_\ell\}$.

Lemma Let $\mathcal{F}$ and $\mathcal{G}$ have the property that for every G_i, there is some F_j with $F_j \prec G_i$. Then $forb(m, \mathcal{F}) \leq forb(m, \mathcal{G})$.

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Theorem Let $\mathcal{F}$ be given. Then either there is a constant c with $\mathit{forb}(m, \mathcal{F}) = c$ or $\mathit{forb}(m, \mathcal{F})$ is $\Omega(m)$.

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Theorem Let $\mathcal{F}$ be given. Then either there is a constant c with $forb(m, \mathcal{F}) = c$ or $forb(m, \mathcal{F})$ is $\Omega(m)$.
Proof: We start using $\mathcal{G} = \{I_p, I_p^c, T_p\}$ with p suitably large.
Either we have the property that there is some $F_r \prec I_p$, and some $F_s \prec I_p^c$ and some $F_t \prec T_p$ in which case
$forb(m, \mathcal{F}) \leq forb(m, \{I_p, I_p^c, T_p\}) = O(1)$
or
without loss of generality we have $F_j \not\prec I_p$ for all j and hence $I_m \in \text{Avoid}(m, \mathcal{F})$ and so $forb(m, \mathcal{F})$ is $\Omega(m)$.

A pair of Configurations with quadratic bounds

e.g. $F_2(1,2,2,1) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \notin I \times I^c.$

$$\underset{I_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \times \underset{I_3^c}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

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$I_{m/2} \times I^c_{m/2}$ is an $m \times m^2/4$ simple matrix avoiding $F_2(1,2,2,1)$, so $forb(m, F_2(1,2,2,1))$ is $\Omega(m^2)$.

(A, Ferguson, Sali 01 $forb(m, F_2(1,2,2,1)) = \lfloor \frac{m^2}{4} \rfloor + \binom{m}{1} + \binom{m}{0}$)

A pair of Configurations with quadratic bounds

e.g. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \notin T \times T.$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{T_3} \times \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{T_3} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

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$T_{m/2} \times T_{m/2}$ is an $m \times m^2/4$ simple matrix avoiding I_3,
so $forb(m, I_3)$ is $\Omega(m^2)$.
$\left(forb(m, I_3) = \binom{m}{2} + \binom{m}{1} + \binom{m}{0}\right)$

Forbidden Families can pass Conjecture

By considering the construction $I \times I^c$ that avoids $F_2(1,2,2,1)$ and the constructions $I^c \times I^c$ or $I^c \times T$ or $T \times T$ that avoids I_3, we note that we have only linear obvious constructions (I_m^c or T_m) that avoid both $F_2(1,2,2,1)$ and I_3. We are led to the following:
Theorem $forb(m, \{I_3, F_2(1,2,2,1)\})$ is $\Theta(m)$.

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We can extend the argument quite far:
Theorem $\mathit{forb}(m, \{t \cdot I_k, F_2(1,t,t,1)\})$ is $\Theta(m)$.

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We studied the 9 'minimal' configurations that have quadratic bounds and were able to verify the predictions of the conjecture for all pairs. A variety of proofs of the upper bounds were employed.

Balogh Bollobás extended

Using our standard induction one can prove the following.

Theorem Let k be given. Then $forb(m, \{2 \cdot I_k, 2 \cdot I_k^c, 2 \cdot T_k\})$ is $\Theta(m)$.

$I_m \in \text{Avoid}(m, \{2 \cdot I_k, 2 \cdot I_k^c, 2 \cdot T_k\})$.

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Theorem Let k, t be given. Then $forb(m, \{t \cdot I_k, t \cdot I_k^c, t \cdot T_k\})$ is $\Theta(m)$.

An unusual Bound

Theorem (A,Koch,Raggi,Sali 12) *forb*$(m, \{T_2 \times T_2, I_2 \times I_2\})$ *is* $\Theta(m^{3/2})$.

$$T_2 \times T_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \; I_2 \times I_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad (= C_4)$$

We showed initially that *forb*$(m, \{T_2 \times T_2, T_2 \times I_2, I_2 \times I_2\})$ is $\Theta(m^{3/2})$ but Christina Koch realized that we ought to be able to drop $T_2 \times I_2$ and we were able to redo the proof (which simplified slightly!).


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Induction

Let A be an $m \times forb(m, \mathcal{F})$ simple matrix with no configuration in $\mathcal{F} = \{T_2 \times T_2, I_2 \times I_2\}$. We can select a row r and reorder rows and columns to obtain

$$A = \begin{array}{c} \text{row } r \\ \\ \end{array} \left[\begin{array}{cccccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & & C_r & C_r & & D_r \end{array} \right].$$

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To show $\|A\|$ is $O(m^{3/2})$ it would suffice to show $\|C_r\|$ is $O(m^{1/2})$ for some choice of r. Our proof shows that assuming $\|C_r\| > 20m^{1/2}$ for all choices r results in a contradiction. In particular, associated with C_r is a set of rows $S(r)$ with $S(r) \geq 5m^{1/2}$. We let $S(r) = \{r_1, r_2, r_3, \ldots\}$. After some work we show that $|S(r_i) \cap S(r_j)| \leq 5$. Then we have
$|S(r_1) \cup S(r_2) \cup S(r_3) \cup \cdots|$
$= |S(r_1)| + |S(r_2) \backslash S(r_1)| + |S(r_3) \backslash (S(r_1) \cup S(r_2))| + \cdots$
$= 5m^{1/2} + (5m^{1/2} - 5) + (5m^{1/2} - 10) + \cdots > m$! ! !

Thanks to Ryan Martin for the invite!
Great to see Ames, Iowa.