Berge Hypergraphs

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October 4, 2016

The results in this paper come from work done this summer with Richard Anstee. The motivation for studying this problem came from Anstee's work in forbidden configurations as well as Daniél Gerbner and Corey Palmers' paper *Extremal Results for Berge-Hypergraphs*.

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- Let $\mathbf{1}_a\mathbf{0}_b$ denote the column of a 1's above b 0's. If a of b are 0 we write $\mathbf{1}_a$ or $\mathbf{0}_b$ instead.
- Let K_k denote the (0,1) k-rowed matrix containing all distinct columns. eg.

$$K_3 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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- Define the extremal function $\text{Bh}(m, F)$ as

$$\text{Bh}(m, F) = \max_{A}\{||A|| : A \in \text{BAvoid}(m, F)\}.$$

Berge Hypergraph Example

$$F = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

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$$F = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} \color{red}{1} & 0 & \color{red}{1} & \color{red}{1} \\ \color{red}{1} & 1 & \color{red}{0} & \color{red}{1} \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \leq \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$F \lll A$$

Example $\mathrm{Bh}(m, I_k)$

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Let k be given and assume $m \geq k - 1$. Then $\mathrm{Bh}(m, I_k) = 2^{k-1}$.

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Proof:

We prove the upper bound, $\mathrm{Bh}(m, I_k) \leq 2^{k-1}$ by induction on k.

Base Case: Let $k = 1$, then $I_1 = [1]$ and $A \in \mathrm{BAvoid}(m, [1])$ can only be the column of zeros. So $\mathrm{BAvoid}(m, I_1) = 1 = 2^0$.

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Induction step: Assume $A \in \mathrm{BAvoid}(m, I_k)$ and let B be the matrix A with rows of 0's removed. If $||B|| \leq 2^{k-2}$ we are done so assume $||B|| > 2^{k-2}$. Also assume B has atleast k rows. Then $I_{k-1} \ll B$, so permute B to the form

$$B = \left[\begin{array}{c|c} C & D \\ \hline E & G \end{array}\right]$$

where $I_{k-1} \ll E$ and E is $(k-1) \times (k-1)$.

Example BAvoid(m, I_k) (cont.)

$$B = \left[\begin{array}{c|c} C & D \\ \hline E & G \end{array}\right]$$

Note that D is the matrix of zero's so G must be simple. Also note that since B has no nonzero rows, C has a 1. Therefore $I_{k-1} \not\prec G$. By the induction assumption $||G|| \leq 2^{k-2}$ and so $||B|| = ||E|| + ||G|| = k - 1 + 2^{k-2} \leq 2^{k-1}$. This proves $\text{Bh}(m, F) \leq 2^{k-1}$.

Example BAvoid(m, I_k) (cont.)

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The construction $K_{k-1} \times \mathbf{0}_{m-(k-1)}$ is in BAvoid(m, I_k) and has 2^{k-1} columns. Thus $\text{Bh}(m, F) = 2^{k-1}$. ∎

General Results

- Complete asymptotic classification of $\text{Bh}(m, F)$ for all $1, 2, 3, 4$-rowed F.

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$$I_1 \times I_2 \times I_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

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- Asymptotic classification of $\mathrm{Bh}(m, F)$ where F is the vertex-edge incidence matrix of a forest.

Downsets

We use a shifting operation $T_i(A)$ on A where we remove all 1's on row i that do not create a repeated column by their removal. This operation preserves the number of columns and simplicity of the matrix. Also if F is not a Berge hypergraph of A then it is not a Berge hypergraph of $T_i(A)$. We apply $T_m(T_{m-1}(\cdots T_1(A)\cdots))$ until we can no longer remove 1's. If we interpret the resulting matrix $T(A)$ as a set system $\mathcal{S}$ then it is a downset . That is to say, if $S \in \mathcal{S}$ and $S' \subset S$, then $S' \in \mathcal{S}$. For the matrix $T(A)$, if a columns has 1's on rows $r_1, r_2, \ldots, r_t$, then K_t is contained on those t rows.

Conclusion: if $A \in \text{BAvoid}(m, F)$, we can assume A has the downset property!

Shifting Example

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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$$\to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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$$\begin{array}{c} \\ \\ \\ \\ \\ \rightarrow \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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Graph Theory Machinery

Theorem

Let G be a simple graph where each vertex has degree greater than or equal to $k-1$. Then G contains every tree on k vertices as a subgraph.

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eg.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

F is a tree

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Proof:

Let $A \in \mathrm{BAvoid}(m, F)$ and assume A is a downset. For each row r of A with column sum 2^{k-2} or less remove that row and the columns of A with 1's on row r. This corresponds to at most $2^{k-2}m$ column deletions. Any rows left in B have row sum strictly larger than 2^{k-2}. Note that this implies that B has k or more rows since K_{k-1} has row sum 2^{k-2}. Consider the submatrix B_q of B formed by taking the columns with 1's on row q and taking every row but row q. B_q is simple with $||B_q|| > 2^{k-2}$ and therefore contains I_{k-1}. Therefore the vertex q in $G(B)$ will have degree $k-1$ or greater.

F is a tree (cont.)

By the theorem, T is a subgraph in $G(B)$ and therefore F is in the downset of B. Since the downset of B is in A, F is in A. This contradicts the hypothesis so we conclude that A has fewer than $2^{k-2}m$ rows.

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The lower bound follows from the construction I_m. ∎

Avoiding the Complete Bipartite Graph

Let $K_{s,t}$ denote the complete bipartite graph on s and t vertices. The vertex-edge incidince matrix of $K_{s,t}$ is $I_s \times I_t$. We use the following theorems to prove results about $\mathrm{Bh}(m, I_s \times I_t)$.

Theorem

W. G. Brown (1966)
For $t \geq 2$, $\mathrm{ex}(m, K_{2,t})$ is $\Theta(m^{\frac{3}{2}})$.

Theorem

N. Alon, L. Rónyai, T. Szabó (1999)
For $t \geq 3$, $\mathrm{ex}(m, K_{3,t})$ is $\Theta(m^{\frac{5}{3}})$.

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For $t \geq 3$, $\mathrm{Bh}(m, I_3 \times I_t)$ *is* $\Theta(m^2)$.

Column Sum Restriction

For general $I_s \times I_t$, we consider matrices with column sums $1, 2, \ldots, s$. Suppose $A \in \text{BAvoid}(m, F)$ has a column α with column sum s. The number of columns β_i with $\beta_i > \alpha$ is bounded by 2^{t-1}.

$$
\begin{array}{c}
 \\ s \\ \\ \\ \\
\end{array}
\begin{array}{c}
\begin{array}{ccccc} \alpha & \beta_1 & \beta_2 & 2^{t-1}+1 & \beta_n \end{array} \\
\left[\begin{array}{ccccc}
1 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 1 \\
 & & & & \\
 & & B & & \\
 & & & &
\end{array}\right]
\end{array}
$$

Column Sum Restriction (cont.)

Note that B is a simple matrix with $||B|| > \mathrm{Bh}(m, I_t)$ so $I_t \lll B$. We apply the downset idea and note that we can find $I_s \times I_t$ in A.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

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$$
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots \\
1 & 1 & \cdots & 1 \\
1 & & & \\
 & 1 & & \\
 & & \ddots & \\
 & & & 1
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\
1 & & & 1 & & & 1 & & \\
 & \ddots & & & \ddots & & & \cdots & \\
 & & 1 & & & 1 & & & 1
\end{bmatrix}
$$

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$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\ 1 & & & 1 & & & 1 & & \\ & \ddots & & & \ddots & & & \cdots & \\ & & 1 & & & 1 & & & 1 \end{bmatrix}$$

We therefore restrict ourselves to considering the columns of column sum s or less since the number of columns of larger column sum is bounded by a constant times the number of columns of column sum s.

$\mathrm{Bh}(m, I_2 \times I_t)$

Theorem

Let $F = I_2 \times I_t$ *be the vertex-edge incidence matrix of the complete bipartite graph* $K_{2,t}$*. Then* $\mathrm{Bh}(m, F)$ *is* $\Theta(\mathrm{ex}(m, K_{2,t}))$ *which is* $\Theta(m^{\frac{3}{2}})$

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Proof: Let $A \in \mathrm{BAvoid}(m, I_2 \times I_t)$ and assume A has column sums at most 2. The number of columns of column sum 0 or 1 is bounded by $m + 1$ and the number of columns of column sum 2 is bounded by $2^{t-1}\mathrm{ex}(m, K_{2,t})$. It is known that $\mathrm{ex}(m, K_{2,t})$ is $\Theta(m^{\frac{3}{2}})$. Thus $\mathrm{Bh}(m, I_2 \times I_t) \leq 2^{t-1}\mathrm{ex}(m, K_{2,t}) + m + 1$ which is $O(m^{\frac{3}{2}})$.

It follows from the existence of a graph with $\Theta(m^{\frac{3}{2}})$ edges that we can take the corresponding vertex-edge incidence matrix and get the lower bound. Therefore $\mathrm{Bh}(m, I_2 \times I_t)$ is $\Theta(m^{\frac{3}{2}})$. ■

Graph reduction

We cannot use the same approach to determine $\mathrm{Bh}(m, I_3 \times I_t)$ since we must consider edges of size 3. However, we can still reduce $\mathrm{Bh}(m, I_3 \times I_t)$ to a graph theory problem. Let $A \in \mathrm{BAvoid}(m, I_3 \times I_t)$ and let A have the downset property. If A has a column of sum 3 on rows i, j, k, then the vertices i, j, k in $G(A)$ have a triangle. Therefore the number of columns of sum 3 in A is bounded by $\mathrm{ex}(m, K_3, K_{s,t})$. Conversely, we can show that if we have a triangle on rows i, j, k of A, then we can have a column with 1's on those rows. Suppose that on rows i, j, k of $A \in \mathrm{BAvoid}(m, I_3 \times I_t)$ we have a triangle K_3. Append the column α with 1's on i, j, k and 0's elsewhere. Call the new matrix B.

Graph reduction (cont)

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & \alpha \end{array} \\ \begin{array}{c} i \\ j \\ k \end{array} & \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Now suppose the forbidden object has been formed in B. Since it was not in A, column α must be part of the submatrix. Furthermore, since there are two 1's in $I_3 \times I_t$ two of rows i, j, k must also be part of the submatrix. Suppose without loss of generality, that those rows are i, j. We note that column a can not be in the submatrix since that would form a 2×2 submatrix of 1's. However, that implies that we could equivalently take a instead of α in the submatrix. Therefore the forbidden object is in A, a contradiction. We conclude that the forbidden object is not in B.

Bh$(m, I_3 \times I_t)$ Upper bound

Lemma

N. Alon, C. Shikhelman (2015)
For any fixed $s \geq 2$ *and* $t \geq (s-1)! + 1, \mathrm{ex}(m, K_3, K_{s,t})$ *is* $\Theta(m^{3-\frac{3}{s}})$.

Theorem

For $t \geq 3$, $\mathrm{Bh}(m, I_3 \times I_t)$ *is* $\Theta(m^2)$.

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Proof: We consider columns of column sum 3 or less. The number of columns with column sum 2 or less is bounded by $\binom{m}{2} + \binom{m}{1} + \binom{m}{0}$. As we showed before, the number of columns of column sum 3 is bounded by $\mathrm{ex}(m, K_3, K_{3,t})$ which is $\Theta(m^2)$. Therefore we have that $\mathrm{Bh}(m, I_3 \times I_t)$ is $O(m^2)$.

$\mathrm{Bh}(m, I_3 \times I_t)$ Lower bound

For the lower bound we take the construction, G, used in the lemma and construct a matrix with a column of column sum 3 on rows i, j, k if vertices i, j, k of G have a triangle. As we showed, this new matrix avoids $I_3 \times I_t$, is simple, and has $\Theta(m^2)$ columns. Therefore $\mathrm{Bh}(m, I_3 \times I_t)$ is $\Theta(m^2)$ ∎

General $I_s \times I_t$

Although the bounds we have found are for $s = 2$ and $s = 3$, our methods generalize to $I_s \times I_t$. For any s and $t \geq s$, we have that

$$\mathrm{Bh}(m, I_s \times I_t) \text{ is } \Theta\left(\sum_{i=0}^{s} \mathrm{ex}(m, K_i, K_{s,t})\right)$$

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Alon and Shikhelman's work is particularly relevant as can be seen by the following theorem.

Theorem

For any fixed $r, s \geq 2r - 2$, and $t \geq (s-1)! + 1$. Then,

$$\mathrm{ex}(m, K_r, K_{s,t}) \geq \left(\frac{1}{r!} + o(1)\right) m^{r - \frac{r(r-1)}{s}}.$$

Thanks for listening!