Forbidden Configurations: Progress on a Conjecture

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Definition We say that a matrix A is *simple* if it is a (0,1)-matrix with no repeated columns.

i.e. if A is m-rowed then A is the incidence matrix of some family $\mathcal{A}$ of subsets of $[m] = \{1, 2, \ldots, m\}$.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{A} = \Big\{\emptyset, \{1, 2, 4\}, \{1, 4\}, \{1, 2\}, \{1, 2, 3\}, \{1, 3\}\Big\}$$

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Definition We define $|A|$ to be the number of columns in A.

$$|A| = 6$$

Definition Given a matrix F, we say that A has F as a *configuration* if there is a submatrix of A which is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad A = \left[\begin{array}{cccccc} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \;\; 0 \;\; 1} & & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \;\; 1 \;\; 0} & & & 0 & \boxed{0} \end{array}\right]$$

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We consider the property of forbidding a configuration F in A for which we say F is a *forbidden configuration* in A.

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forb(m, F)= $\max\{|A| : A$ m-rowed simple, no configuration $F\}$

Thus if A is any $m \times (\mathit{forb}(m, F) + 1)$ simple matrix then A contains the configuration F.

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For example, $\quad \mathit{forb}(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = m + 1.$

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible columns on k rows.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1})$$

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A Product Construction

The building blocks of our product constructions are I, I^c and T:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin I, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin I^c, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin T$$

Definition Given an $m_1 \times n_1$ matrix A and a $m_2 \times n_2$ matrix B we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of A on top of a column of B. If A, B are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \times \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix} = \begin{bmatrix}1&1&1&0&0&0&0&0&0\\0&0&0&1&1&1&0&0&0\\0&0&0&0&0&0&1&1&1\\1&1&1&1&1&1&1&1&1\\0&1&1&0&1&1&0&1&1\\0&0&1&0&0&1&0&0&1\end{bmatrix}$$

Given p simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the p-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.

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Examples

$$[01] \times [01] = K_2$$

$$\overbrace{[01] \times [01] \times \cdots \times [01]}^{k} = K_k$$

$$I_2 \times I_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \approx C_4$$

$I_{m/2} \times I_{m/2}$ is vertex-edge incidence matrix of $K_{m/2,m/2}$

The Conjecture

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Definition Let $x(F)$ denote the largest p such that there is a p-fold product which does not contain F as a configuration where the p-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

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The conjecture has been verified for $k \times \ell$ F where $k = 2$ (A, Griggs, Sali 97) and $k = 3$ (A, Sali 05) and $l = 2$ (A, Keevash 06) and for k-rowed F with bounds $\Theta(m^{k-1})$ or $\Theta(m^k)$ (A, Fleming 10) plus other cases.

In order for a 4-rowed F to have $forb(m, F)$ be quadratic in m, the associated simple matrix must have a quadratic bound. Using a result of A and Fleming, there are three simple column-maximal 4-rowed F for which $forb(m, F)$ is quadratic. Here is one example:

$$F_8 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

How can we repeat columns in F_8 and still have a quadratic bound? We note that repeating either the column of sum 1 or the column of sum 3 will result in a cubic lower bound. Thus we only consider taking multiple copies of the columns of sum 2.

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$$F_8(t) = \left[\begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{matrix} \; t \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}\right]$$

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Theorem (A, Raggi, Sali 09) Let t be given. Then $forb(m, F_8(t))$ is $O(m^2)$. Moreover $F_8(t)$ is a boundary case, namely for any column α not already present t times in $F_8(t)$, then $forb(m, [F_8(t)|\alpha])$ is $\Omega(m^3)$.

The proof is currently a rather complicated induction with some directed graph arguments. For each α there are $\Omega(m^3)$ product constructions avoiding $[F_8(t)|\alpha]$.

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There are two other 4-rowed F to consider which should have quadratic bounds. This would verify the conjecture for $k = 4$.

Let

$$F(t) = t \cdot \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{bmatrix}$$

Problem *Let t be given. Show that forb$(m, F(t))$ is $O(m^2)$.*

Let

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Problem *Let t be given. Show that forb$(m, F(t))$ is $O(m^2)$.*

The above is true for both $t = 1$ and $t = 2$. The above problem follows from the general conjecture. Solving this problem would no doubt enable the proof of quadratic bounds for the two other 4-rowed F

5×6 Simple Configuration with Quadratic bound

The Conjecture predicts nine 5-rowed simple matrices F which are boundary cases, namely $\mathit{forb}(m, F)$ is $O(m^2)$ and for any column α we have $\mathit{forb}(m, [F|\alpha])$ being $\Omega(m^3)$. Such F happen all to be 5×6 simple matrices and we have handled the following case.

$$F_7 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Theorem (A, Raggi, Sali) $\mathit{forb}(m, F_7)$ is $O(m^2)$.
The proof is currently a rather complicated induction.

Induction

Let A be an $m \times forb(m, F_7)$ simple matrix with no configuration F_7. We can select a row r and reorder rows and columns to obtain

$$A = \begin{array}{c} \text{row } r \\ \\ \end{array} \left[\begin{array}{cccccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ \color{red}{B_r} & & \color{red}{C_r} & \color{red}{C_r} & & \color{red}{D_r} \end{array}\right].$$

Now $[B_r C_r D_r]$ is an $(m-1)$-rowed simple matrix with no configuration F_7. Also C_r is an $(m-1)$-rowed simple matrix with no configurations in $\mathcal{F}$ where $\mathcal{F}$ is derived from F_7.

Then $|A| = forb(m, F) \leq forb(m-1, F) + |C_r|$.

To show $|A|$ is quadratic it would suffice to show $|C_r|$ is linear for some choice of r.

Repeated Induction

Let C_r be an $(m-1)$-rowed simple matrix with no configuration in $\mathcal{F}$. We can select a row s and reorder rows and columns to obtain

$$C_r = \begin{matrix}\text{row } s\\ \\\end{matrix}\begin{bmatrix}0 & \cdots & 0 & 1 & \cdots & 1\\ E_s & & G_s & G_s & & H_s\end{bmatrix}.$$

To show $|C_r|$ is linear it would suffice to show $|G_s|$ is bounded by a constant for some choice of s. Our proof shows that assuming $|G_s| > 8$ for all choices s results in a contradiction.
This repeated induction is used to show that $forb(m, F_7)$ is $O(m^2)$.

An unusual Bound

Theorem (A,Raggi,Sali) $forb(m, \{T_2 \times T_2, T_2 \times I_2, I_2 \times I_2\})$ is $\Theta(m^{3/2})$.

$$T_2 \times T_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \; T_2 \times I_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$I_2 \times I_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Induction

Let A be an $m \times forb(m, \mathcal{F})$ simple matrix with no configuration in $\mathcal{F}$. We can select a row r and reorder rows and columns to obtain

$$A = \begin{array}{c} \text{row } r \\ \end{array} \left[\begin{array}{cccccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & & C_r & C_r & & D_r \end{array} \right].$$

To show $|A|$ is $O(m^{3/2})$ it would suffice to show $|C_r|$ is $O(m^{1/2})$ for some choice of r. Our proof shows that assuming $|C_r| > 16m^{1/2}$ for all choices r results in a contradiction.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Definition For an m-rowed matrix P, we define $f(F, P) = \max\{|A| : A \text{ an } m-\text{rowed submatrix of } P\text{, no configuration } F\}$.

For example, with K_m denoting the $m \times 2^m$ simple matrix, then $forb(m, F) = f(F, K_m)$.

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The first result follows from the Marcus and Tardos (05) results on forbidden patterns.

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The same proof techniques will show

Theorem $f(T_k \times T_k, T_{m/2} \times T_{m/2})$ is $O(m)$

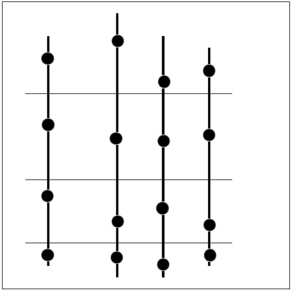
Theorem $f(I_3 \times T_3, I_{m/2} \times T_{m/2})$ is $O(m)$

The same proof techniques will show

Theorem $f(T_k \times T_k, T_{m/2} \times T_{m/2})$ is $O(m)$

Theorem $f(I_3 \times T_3, I_{m/2} \times T_{m/2})$ is $O(m)$

Problem Can you show $f(I_4 \times T_4, I_{m/2} \times T_{m/2})$ is $O(m)$?



THANKS to the session organizers! Hope you have enjoyed Vancouver.