Forbidden Configurations: A Survey

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Introduction

Forbidden configurations are first described as a problem area in a 1985 paper. The subsequent work has involved a number of coauthors: Farzin Barekat, Laura Dunwoody, Ron Ferguson, Balin Fleming, Zoltan Füredi, Jerry Griggs, Nima Kamoosi, Steven Karp, Peter Keevash and Attila Sali but there are works of other authors (some much older, some recent) impinging on this problem as well. For example, the definition of VC-dimension uses a forbidden configuration.

Survey at www.math.ubc.ca/∼anstee

$$[m] = \{1, 2, \ldots, m\}$$
$$2^{[m]} = \{A : A \subseteq [m]\}$$
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Theorem *If* $\mathcal{F} \subseteq 2^{[m]}$, *then*

$$|\mathcal{F}| \leq 2^m.$$

Definition We say $\mathcal{F} \subseteq 2^{[m]}$ is intersecting if for every pair $A, B \in \mathcal{F}$, we have $|A \cap B| \geq 1$.

Theorem *If* $\mathcal{F} \subseteq 2^{[m]}$ *and* $\mathcal{F}$ *is intersecting, then*

$$|\mathcal{F}| \leq 2^{m-1}.$$

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i.e. if A is m-rowed then A is the incidence matrix of some $\mathcal{F} \subseteq 2^{[m]}$.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathcal{F} = \left\{\emptyset, \{2\}, \{3\}, \{1,3\}, \{1,2,3\}\right\}$$

Definition Given a matrix F, we say that A has F as a *configuration* if there is a submatrix of A which is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} = A$$

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We consider the property of forbidding a configuration F in A for which we say F is a *forbidden configuration* in A.

Definition Let $\text{forb}(m, F)$ be the largest function of m and F so that there exist a $m \times \text{forb}(m, F)$ simple matrix with *no* configuration F. Thus if A is any $m \times (\text{forb}(m, F) + 1)$ simple matrix then A contains F as a configuration.

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For example, $\text{forb}(m, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) = 2, \quad \text{forb}(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = m + 1.$

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible columns on k rows (i.e. incidence matrix of $2^{[k]}$).

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots \binom{m}{0} = \Theta(m^{k-1})$$

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Theorem (Füredi 83). *Let F be a $k \times l$ matrix. Then $forb(m, F) = O(m^k)$*

Two interesting examples

$$\text{Let} \qquad F_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\text{forb}(m, F_1) = 2m, \qquad \text{forb}(m, F_2) = \left\lfloor \frac{m^2}{4} \right\rfloor + m + 1$$

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Problem *What drives the asymptotics of forb(m, F)? What structures in F are important?*

A Product Construction

The building blocks of our product constructions are I, I^c and T:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin I, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin I^c, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin T$$

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Note that $\text{forb}(m, \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \text{forb}(m, \begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \text{forb}(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = m + 1$

Definition Given an $m_1 \times n_1$ matrix A and a $m_2 \times n_2$ matrix B we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of A on top of a column of B. If A, B are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Given p simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the p-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.

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The Conjecture

Definition Let $x(F)$ denote the largest p such that there is a p-fold product which does not contain F as a configuration where the p-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

Thus $x(F) + 1$ is the smallest value of p such that F is a configuration in every p-fold product $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

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The conjecture has been verified for $k \times l$ F where $k = 2$ (A, Griggs, Sali 97) and $k = 3$ (A, Sali 05) and $l = 2$ (A, Keevash 06) and for k-rowed F with bounds $\Theta(m^{k-1})$ or $\Theta(m^k)$ plus other cases.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71) *forb*(m, K_k) *is* $\Theta(m^{k-1})$

Let $E_1 = \begin{bmatrix}1\\1\end{bmatrix}, E_2 = \begin{bmatrix}0\\0\end{bmatrix}, E_3 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}.$

Theorem (A, Fleming) *Let* F *be a* $k \times l$ *simple matrix such that there is a pair of rows with no configuration* E_1 *and there is a pair of rows with no configuration* E_2 *and there is a pair of rows with no configuration* E_3*. Then forb*(m, F) *is* $O(m^{k-2})$.

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Note that $F_1 = \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 1 & 1\\0 & 0 & 0 & 1\end{bmatrix}$ has no E_1 on rows 1,3, no E_2 on rows 1,2 and no E_3 on rows 2,3. Thus forb(m, F_1) is $O(m)$.

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Theorem (A, Fleming) *Let E be given with $E \in \{E_1, E_2, E_3\}$. Let F be a $k \times l$ simple matrix with the property that every pair of rows contains the configuration E. Then forb$(m, F) = \Theta(m^{k-1})$.*

$$F_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$ has E_3 on rows 1,2.

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$$F_2 = \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\0 & 0 & 1 & 1\end{bmatrix}$$ has E_3 on rows 2,3.

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Note that F_2 has E_3 on every pair of rows hence forb(m, F_2) is $\Theta(m^2)$ (A, Griggs, Sali 97).

In particular, this means $F_2 \notin T \times T$ which is the construction to achieve the bound.

Definition Let $t \cdot M$ be the matrix $[M\,M \cdots M]$ consisting of t copies of M placed side by side.

Theorem (A, Füredi 86)

$$forb(m, t\cdot K_k) = \frac{t-2}{k+1}\binom{m}{k}(1+o(1)) + \binom{m}{k} + \binom{m}{k-1} + \cdots \binom{m}{0}$$

Let B be a $k \times (k+1)$ matrix which has one column of each column sum. Given two matrices C, D, let $C \backslash D$ denote the matrix obtained from C by deleting any columns of D that are in C (i.e. set difference). Let

$F_B(t) = [K_k | t \cdot [K_k \backslash B]]$.

Theorem (A, Griggs, Sali 97, A, Sali 05,
A, Fleming, Füredi, Sali 05)
forb$(m, F_B(t))$ *is* $\Theta(m^{k-1})$.

The difficult problem here was the bound although induction works.

Let D be the $k \times (2^k - 2^{k-2} - 1)$ simple matrix with all columns of sum at least 1 that do not simultaneously have 1's in rows 1 and 2. We take $F_D(t) = [\mathbf{0}_k \; (t+1) \cdot D]$ which for $k = 4$ becomes

$$F_D(t) = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \; (t+1) \cdot \left[\begin{array}{ccccccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{array} \right] \right]$$

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The argument used standard results for directed graphs, *indicator polynomials* and a linear algebra rank argument

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The argument used standard results for directed graphs, *indicator polynomials* and a linear algebra rank argument

Theorem *Let k be given and assume F is a k-rowed configuration which is not a configuration in $F_B(t)$ for any choice of B as a $k \times (k+1)$ simple matrix with one column of each column sum and not in $F_D(t)$, for any t. Then forb(m, F) is $\Theta(m^k)$.*

A 2-design $S_\lambda(2,3,v)$ consists of $\frac{\lambda}{3}\binom{v}{2}$ triples from $[v]=\{1,2,\ldots,v\}$ such that for each pair $i,j\in\binom{[v]}{2}$, there are exactly λ triples containing i,j. If we encode the triple system as a v-rowed (0,1)-matrix A such that the columns are the incidence vectors of the triples, then A has no $(\lambda+1)\times 2$ submatrix of 1's.

Designs and Forbidden Configurations

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Remark If A is a $v\times n$ (0,1)-matrix with column sums 3 and A has no $(\lambda+1)\times 2$ submatrix of 1's then $n\leq\frac{\lambda}{3}\binom{m}{2}$ with equality if and only if the columns of A correspond to the triples of a 2-design $S_{\lambda}(2,3,v)$.

Theorem (A, Barekat) Let λ and v be given integers. There exists an M so that for $v > M$, if A is an $v \times n$ (0,1)-matrix with column sums in $\{3, 4, \ldots, v-1\}$ and A has no $(\lambda + 1) \times 3$ configuration

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

then

$$n \leq \frac{\lambda}{3}\binom{v}{2}$$

and we have equality if and only if the columns of A correspond to the triples of a 2-design $S_\lambda(2, 3, v)$.

Theorem (A, Barekat) Let λ and v be given integers. There exists an M so that for $v > M$, if A is an $v \times n$ (0,1)-matrix with column sums in $\{3, 4, \ldots, v-3\}$ and A has no $(\lambda+1) \times 4$ configuration

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

then

$$n \leq \frac{\lambda}{3}\binom{v}{2}$$

with equality only if there are positive integers a, b with $a + b = \lambda$ and there are $\frac{a}{3}\binom{v}{2}$ columns of A of column sum 3 corresponding to the triples of a 2-design $S_a(2, 3, v)$ and there are $\frac{b}{3}\binom{v}{2}$ columns of A of column sum $v - 3$ corresponding to $(v-3)$ - sets whose complements (in $[v]$) corresponding to the triples of a 2-design $S_b(2, 3, v)$.

Exact Bounds

A, Barekat 08

Configuration F	Exact Bound forb(m, F)
$\overbrace{\begin{bmatrix} 1\,1\cdots 1 \\ 1\,1\cdots 1 \\ 0\,0\cdots 0 \end{bmatrix}}^{p}$	$\frac{p+1}{3}\binom{m}{2}+\binom{m}{1}+2\binom{m}{0}$ for m large, $m\equiv 1,3 (\text{mod} 6)$
$\overbrace{\begin{bmatrix} 1\,1\cdots 1 \\ 1\,1\cdots 1 \\ 0\,0\cdots 0 \\ 0\,0\cdots 0 \end{bmatrix}}^{p}$	$\frac{p+3}{3}\binom{m}{2}+2\binom{m}{1}+2\binom{m}{0}$ for m large, $m\equiv 1,3 (\text{mod} 6)$

Exact Bounds

A, Griggs, Sali 97, A, Ferguson, Sali 01, A, Kamoosi 07
A, Barekat, Sali 08, A, Barekat 08, A, Karp 09

Configuration F	Exact Bound forb(m, F)
$\begin{bmatrix}0\\1\end{bmatrix}$	2
$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	$m+2$
$\begin{bmatrix}0&0&0\\1&1&1\end{bmatrix}$	$2m+2$
$\begin{bmatrix}0&0&0&0\\1&1&1&1\end{bmatrix}$	$\left\lfloor \frac{5m}{2}\right\rfloor+2$
$q\cdot\begin{bmatrix}0\\1\end{bmatrix}$	$\left\lfloor \frac{(q+1)m}{2}\right\rfloor+2,\quad$ for m large

Exact Bounds

Configuration F	Exact Bound forb(m, F)
$\begin{bmatrix}1&0&0\\0&1&1\end{bmatrix}$	$\left\lfloor \frac{3m}{2} \right\rfloor + 1$
$\begin{bmatrix}1&0&0&0\\0&1&1&1\end{bmatrix}$	$\left\lfloor \frac{7m}{3} \right\rfloor + 1$
$\begin{bmatrix}1&0&0&0&0\\0&1&1&1&1\end{bmatrix}$	$\left\lfloor \frac{11m}{4} \right\rfloor + 1$
$\begin{bmatrix}1&0&0&0&0&0\\0&1&1&1&1&1\end{bmatrix}$	$\left\lfloor \frac{15m}{4} \right\rfloor + 1$

Exact Bounds

Configuration F	Exact Bound forb(m, F)
$\begin{bmatrix}1&1&0&0&0\\0&0&1&1&1\end{bmatrix}$	$\lfloor \frac{8m}{3} \rfloor$
$\begin{bmatrix}1&1&0&0&0&0\\0&0&1&1&1&1\end{bmatrix}$	$\lfloor \frac{10m}{3} - \frac{4}{3} \rfloor$
$\begin{bmatrix}1&1&0&0&0&0&0\\0&0&1&1&1&1&1\end{bmatrix}$	$4m$
$\begin{bmatrix}\overbrace{1\cdots1}^{p}\overbrace{0\cdots0}^{p}\\0\cdots0\,1\cdots1\end{bmatrix}$	$pm - p + 2$

Exact Bounds

$$F_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Theorem (A, Dunwoody) $\text{forb}(m, F_2) = \lfloor \frac{m^2}{4} \rfloor + m + 1$

Proof: The proof technique is that of shifting, popularized by Frankl. A paper of Alon 83 using shifting refers to the possibility of such a result.

$k \times 2$ Forbidden Configurations

$$\text{Let } F_{abcd} = \begin{matrix} a\left\{\begin{matrix} \\ \\ \\ \end{matrix}\right. \\ b\left\{\begin{matrix} \\ \\ \\ \end{matrix}\right. \\ c\left\{\begin{matrix} \\ \\ \\ \end{matrix}\right. \\ d\left\{\begin{matrix} \\ \\ \\ \end{matrix}\right. \end{matrix} \begin{bmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

For the purposes of forbidden configurations we may assume that $a \geq d$ and $b \geq c$.

The following result used a difficult ‘stability’ result and the resulting constants in the bounds were unrealistic but the asymptotics are further evidence for the conjecture.

Theorem (A-Keevash 06) *Assume a,b,c,d are given with* $a \geq d$ *and* $b \geq c$*. If* $b > c$ *or* $a, b \geq 1$*, then*

$$forb(m, F_{abcd}) = \Theta(m^{a+b-1}).$$

Also $forb(m, F_{0bb0}) = \Theta(m^b)$ *and* $forb(m, F_{a00d}) = \Theta(m^a)$*.*

It is convenient to define $\mathbf{1}_k\mathbf{0}_\ell$ as the $(k+\ell)\times 1$ column of k 1's on top of ℓ 0's. Then the first column of F_{abcd} is $\mathbf{1}_{a+b}\mathbf{0}_{c+d}$.

Theorem (A, Karp 09) Let $a, b \geq 2$. Then

$$\text{forb}(m, F_{ab01}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_1) = \sum_{j=0}^{a+b-1}\binom{m}{j} + \sum_{j=m}^{m}\binom{m}{j}$$

$$\text{forb}(m, F_{ab10}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_1) = \sum_{j=0}^{a+b-1}\binom{m}{j} + \sum_{j=m}^{m}\binom{m}{j}$$

$$\text{forb}(m, F_{ab11}) = \text{forb}(m, \mathbf{1}_{a+b}\mathbf{0}_2) = \sum_{j=0}^{a+b-1}\binom{m}{j} + \sum_{j=m-1}^{m}\binom{m}{j}$$

$$F_{0220} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Not all $k \times 2$ cases are obvious:

Theorem (A, Barekat, Sali)

$$\text{forb}(m, F_{0220}) = \binom{m}{2} + m - 2$$

$$F_{0220} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Not all $k \times 2$ cases are obvious:

Theorem (A, Barekat, Sali)

$$\text{forb}(m, F_{0220}) = \binom{m}{2} + m - 2$$

Conjecture *forb*$(m, t \cdot F_{0220})$ *is* $O(m^2)$.

The result is true for $t = 2$. The result would follow from the general conjecture

$$F_{2110} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Not all $k \times 2$ cases are obvious:

Theorem *Let c be a positive real number. Let A be an $m \times \left(c\binom{m}{2} + m + 2\right)$ simple matrix with no F_{2110}. Then for some $M > m$, there is an $M \times \left((c + \frac{2}{m(m-1)})\binom{M}{2} + M + 2\right)$ simple matrix with no F_{2110}.*

THANKS FOR THE CHANCE TO VISIT LETHBRIDGE!

Definition We say $\mathcal{F} \subseteq 2^{[m]}$ is *t-intersecting* if for every pair $A, B \in \mathcal{F}$, we have $|A \cap B| \geq t$.

Theorem (Ahlswede and Khachatrian 97)
Complete Intersection Theorem.
Let k, r be given. A maximum sized $(k\text{-}r)$-intersecting k-uniform family $\mathcal{F} \subseteq \binom{[m]}{k}$ is isomorphic to $\mathcal{I}_{r_1,r_2}$ for some choice $r_1 + r_2 = r$ and for some choice $G \subseteq [m]$ where $|G| = k - r_1 + r_2$ where
$\mathcal{I}_{r_1,r_2} = \{A \subseteq \binom{[m]}{k} \,:\, |A \cap G| \geq k - r_1\}$

This generalizes the Erdős-Ko-Rado Theorem (61).

Theorem (A-Keevash 06) Stability Lemma.
Let $\mathcal{F} \subseteq \binom{[m]}{k}$. *Assume that* $\mathcal{F}$ *is* $(k\text{-}r)$*-intersecting and*

$$|\mathcal{F}| \geq (6r)^{5r+7} m^{r-1}.$$

Then $\mathcal{F} \subseteq \mathcal{I}_{r_1,r_2}$ *for some choice* $r_1 + r_2 = r$ *and for some choice* $G \subseteq [m]$ *where* $|G| = k - r_1 + r_2$.

This result is for large intersections; we use it with a fixed r where k can grow with m.