Math 105 Integration Facts Richard Anstee 2016

The common course website contains the definitive information of the facts you need to know.
1. Easy antiderivatives.
You recall differentiating functions and so some antiderivatives are known:

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C \text{ for } n \neq -1$$

$$\int \frac{1}{x} dx = \ln(x) + C$$

$$\int e^x dx = e^x + C$$

$$\int \sin(x) dx = -\cos(x) + C$$

$$\int \cos(x) dx = \sin(x) + C$$

We introduced the inverse trigonometric functions precisely to tackle the following

$$\int \frac{1}{1+x^2} dx = \tan^{-1}(x) + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + C = -\cos^{-1}(x) + C$$

We combine with other rules to find antiderivatives for a variety of functions including polynomials.

$$\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$$

$$\int kf(x) = k \int f(x) dx$$

Note that these rules arise from the similar rules for differentiation.

The chain rule for differentiation gives rise to

$$\int f'(g(x))g'(x) dx = f(g(x)) + C$$

We use this rule very successfully to attack a variety of integrals:

$$\int f(3x-2) dx$$

can be tackled by using the idea that $g(x) = 3x-2$. For convenience we write this as the substitution $u = 3x - 2$ and $du = 3dx$ so that

$$\int f(3x-2) dx = \frac{1}{3} \int f(u) du,$$

where hopefully the second integral is more straightforward.

The product rule for differentiation gives rise the idea of integration by parts.

$$\int \left(f'(x)g(x) + f(x)g'(x) \right) dx = f(x)g(x) + C$$

where we use this as follows

$$\int f'(x)g(x) = f(x)g(x) - \int f(x)g'(x)dx + C$$

We need not write down the C in this equation (although remembering it in most other circumstances is required) because the second integral is an indefinite integral and so it has a built in constant C already.

To aid computations we write this as

$$\int u'vdx = uv - \int uv'dx$$

Amazingly this solves many integrals. Our first applications are to obtain some reductions:

$$\int x^k e^x$$

Let $u = x^k$ and $v' = e^x$ and then $u' = kx^{k-1}$ while $v = e^x$.

$$\int x^k e^x dx = x^k e^x - \int kx^{k-1} e^x dx$$

where the latter integral has a reduced power of x. We can repeat until we get down to

$$\int x^0 e^x dx = e^x + C$$

Sometimes the reduction works right away (in the above example when $k = 1$) or sometimes there is no apparent reduction but we encounter the desired integral again and then get an equation involving the desired integral which we can solve.

$$\int \sec^3(x)dx$$

We use $\sec^3(x) = \sec^2(x)\sec(x)$ and then apply integration by parts with $u' = \sec^2(x)$, $v = \sec(x)$ so that $u = \tan(x)$ and $v' = \tan(x)\sec(x)$ so that

$$\int \sec^3(x)dx = \sec(x)\tan(x) - \int \tan^2(x)\sec(x)dx = \sec(x)\tan(x) - \int (1 + \sec^2(x))\sec(x)dx$$

$$= \sec(x)\tan(x) - \int \sec(x) - \int \sec^3(x)dx$$

We already know that $\int \sec(x)dx = \ln(|\sec(x) + \tan(x)|) + C$ and we shuffle to obtain

$$2\int \sec^3(x)dx = \sec(x)\tan(x) - \ln(|\sec(x) + \tan(x)|) + C$$

so that

$$\int \sec^3(x)dx = \frac{1}{2}\sec(x)\tan(x) - \frac{1}{2}\ln(|\sec(x) + \tan(x)|) + C.$$

This is an example of the reduction formulas given in the book. When I checked on wikipedia they give both the derivation and some applications of this function. Admittedly these examples are unlikely to arise in Economics but the computation of $\int \sin^m(x)\cos^n(x)dx$ and $\int \tan^m(x)\sec^n(x)dx$ are nice examples when we have a complete description how to proceed.

Specialized techniques are useful for trignometric integrals, integrals involving expressions such as $\sqrt{a^2 + x^2}$ or $\sqrt{a^2 - x^2}$ (method of trigonometric substitutions), or integrals involving ratios of polynomials (method of partial fractions). This means you have a wealth of possible solution strategies and so when faced with an given integral you should consider it careful before proceeding. Try a substitution for example.

Finally not all integrals have attractive answers and can only be readily approached by numerical approximations.