MATH 223 Assignment #1 Due Friday September 17 at start of class. Upload to Canvas.

1. Let \( A = \begin{bmatrix} x & 1 \\ 1 & 2 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & 1 \\ x & y \end{bmatrix} \). Determine all \( x, y \) so that \( AB = BA \).

2. Find a \( 2 \times 2 \) matrix \( A \), no entry of which is 0, with \( A^2 = -A \). Note that your first guesses \( A = -I \) or \( A = 0 \) (which would arise from \( A^2 + A = A(A + I) = 0 \)) but both these matrices have 0 entries. Polynomials in matrices behave in different way that polynomials of a single variable.

3. Gavin has a sum of \( s \) dollars in \( t \) bills (not treasury bills!) The bills are either $3 or $5 (Cook Islands?). Give a \( 2 \times 2 \) matrix taking \( x \) (number of $3 bills) and \( y \) (number of $5 bills) and giving as output \( s \) and \( t \).

4. Assume you are given a pair of matrices \( A, B \) which satisfy \( AB = BA \) (We say \( A \) and \( B \) commute). Show that if we set \( C = A^2 + 2A \) and \( D = B^3 + 5I \), then \( CD = DC \). Then try to generalize this in some interesting way, namely find a property so that for matrices \( C, D \) with that certain property, then \( CD = DC \). For example \( C = A^2 + 6A \) and \( D = 3B^3 - 2I \) will also have \( CD = DC \).

5. a) Assume \( A, B \) are \( 2 \times 2 \) invertible matrices so that \( A^{-1} \) and \( B^{-1} \) exist. Show that \( (AB)^{-1} = B^{-1}A^{-1} \).

b) Given
\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]
then define \( A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \),
where \( A^T \) is called the transpose of \( A \). The dot product of two vectors \( x = \begin{bmatrix} a \\ b \end{bmatrix} \), \( y = \begin{bmatrix} c \\ d \end{bmatrix} \) is \( x \cdot y = ac + bd \). Then the \( i, j \) entry of \( AB \) is the dot product of the \( i \)th row of \( A \) and the \( j \)th column of \( B \). Using this idea, show that \( (AB)^T = B^T A^T \). (One could verify \( (AB)^T = B^T A^T \) for two arbitrary \( 2 \times 2 \) matrices \( A, B \) directly but the argument wouldn’t generalize to larger matrices).

6. Find the matrix \( A \) associated with the linear transformation \( T \) that has \( T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \) and \( T \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \).

7. Let \( f_\theta \) be the linear transformation rotating the plane by \( \theta \) counterclockwise around the origin. Let \( A_\theta \) denote the matrix corresponding to the transformation \( f_\theta \). Explain in words (using transformations) why \( A_\theta A_\phi = A_{\theta + \phi} \). Show how you can use this to derive the angle sum formulas for \( \cos(\theta + \phi), \sin(\theta + \phi) \) in terms of \( \cos(\theta), \sin(\theta), \cos(\phi), \sin(\phi) \).

8. Consider the rotation matrix \( A_\theta \) as above. Explain why it has a unique inverse.

9. Consider two nonzero vectors \( x = \begin{bmatrix} a \\ b \end{bmatrix} \), \( y = \begin{bmatrix} c \\ d \end{bmatrix} \), each of unit length. Then there is a \( \theta \) with \( 0 \leq \theta < 2\pi \) so that \( y = A_\theta x \). Use our knowledge of rotation matrices to establish a simple condition on \( a, b, c, d \) so that the angle \( \theta \) satisfies \( 0 < \theta < \pi \). You may assume \( a, b, c, d \) are nonzero, if that assists you.