1. Define $\text{tr} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d$ (read trace for ‘tr’).

   a) Using $A^* = (a + d)I - A$ verify $AA^* = (ad - bc)I$ and verify the Cayley-Hamilton Theorem (at least for $2 \times 2$ matrices):

   $$A^2 - \text{tr}(A)A + \det(A)I = 0$$

   (i.e. A acts as a ‘root’ to the quadratic $\det(A - \lambda I)$ when interpreted as a matrix expression).

   b) Determine conditions on $\text{tr}(A)$ and $\det(A)$ to ensure that $A^2 = -A$ but $A \neq -I, O$. Substitute $-A$ for $A^2$ in the Cayley-Hamilton Theorem to obtain a matrix equation in $A$. First solve for $k$ if $A = kI$ for any $k$ and see what solutions arise. Second assume $A \neq kI$ for any $k$. Then solve for $\text{tr}(A)$ and $\det(A)$.

2. a) Assume we have two nonzero vectors $\mathbf{u}, \mathbf{v}$. Let $M = [\mathbf{u} \ \mathbf{v}]$, the $2 \times 2$ matrix with column 1 being $\mathbf{u}$ and column 2 being $\mathbf{v}$. Show that the $2 \times 2$ matrix $M = [\mathbf{u} \ \mathbf{v}]$ is invertible if and only if $\mathbf{u} \neq p\mathbf{v}$ for any real number $p$.

   b) Assume $A$ is a $2 \times 2$ matrix with two different eigenvalues $\lambda_1 \neq \lambda_2$ and eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ ($A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ and $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$). Show that $\mathbf{v}_1 \neq p\mathbf{v}_2$ for any real number $p$. Thus show that the matrix $M = [\mathbf{v}_1 \ \mathbf{v}_2]$ is invertible and thus show that $A$ is diagonalizable.

3. Let $A_1 = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$. For each of these two matrices, determine the eigenvalues and for each eigenvalue determine an eigenvector. For $A_2$ the eigenvalues are a little more complicated making the computations a little harder. Then give the diagonalization of each matrix; namely an invertible matrix $M$ and a diagonal matrix $D$ with $AM = MD$. (the equation $AM = MD$ is important because it will yield $A = MDM^{-1}$ and $M^{-1}AM = D$ but don’t compute $M^{-1}$ etc.).

4. Recall that $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$.

   Define $e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \cdots$ assuming that the infinite sum makes sense (it does). We are generalizing exponentiation to matrices.

   a) For $D = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, compute $e^D$.

   b) For $B = MDM^{-1}$ where $M = \begin{bmatrix} 2 & 6 \\ 1 & 2 \end{bmatrix}$ and $D$ is as above, compute the entries of $e^B$. (Use our matrix distributive law on an infinite sum; we haven’t justified that this is reasonable but it is OK for you to use it here). This will generalize a) to any diagonalizable matrix.
5. Let \( A \) be a \( 2 \times 2 \) matrix with two different eigenvalues \( \lambda_1, \lambda_2 \) and associated eigenvectors \( \mathbf{v}_1, \mathbf{v}_2 \). Let \( \mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2 \). Assume that \( |\lambda_1| > |\lambda_2| \). Show that
\[
\lim_{n \to \infty} A^n \frac{\mathbf{v}}{\lambda_1^n} = a\mathbf{v}_1
\]
For \( a \neq 0 \) this means that we see the eigenvector \( \mathbf{v}_1 \) appearing in the limit.

6. Review the notes on Fibonacci numbers. Let \( f_1, f_2 \) be two arbitrary integers, not both zero. Consider the sequence \( f_1, f_2, f_3, f_4, \ldots \) where \( f_i = f_{i-1} + f_{i-2} \) for \( i = 3, 4, 5, \ldots \). We wish to show that
\[
\lim_{n \to \infty} \frac{f_n}{f_{n-1}} = \frac{1 + \sqrt{5}}{2}.
\]
(My Linear Algebra instructor Harry F Davis, in 1973, said that he was contacted by a member of the public who had noticed this lovely fact)

Firstly, explain why we can solve for \( c_1, c_2 \) in the vector equation
\[
\begin{bmatrix} f_2 \\ f_1 \end{bmatrix} = c_1 \begin{bmatrix} 1 + \sqrt{5} \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 - \sqrt{5} \\ 2 \end{bmatrix}.
\]
Using our hypothesis that \( f_1, f_2 \) are not both zero, we deduce that \( c_1, c_2 \) are not both zero. Secondly, use our hypothesis that \( f_1, f_2 \) are integers, not both zero, to deduce \( c_1 \neq 0 \). The irrationality of \( \sqrt{5} \) (which you need not prove) combined with \( f_1, f_2 \) being integers is important. Thirdly verify the limit. If you can’t show \( c_1 \neq 0 \) then you can still proceed for a partial solution by assuming \( c_1 \neq 0 \) and then establishing this limit.

Hint: use ideas of question 5.