1. Show that the $2 \times 2$ matrix of 0’s is diagonalizable (I mention this because many students confuse invertibility and diagonalizability; they are not related).

2. Let $A_k$ denote the $2 \times 2$ matrix

$$A_k = \begin{bmatrix} k & 1 \\ -1 & 3 \end{bmatrix}$$

For what values of $k$ is $A_k$ diagonalizable? Namely, for what values of $k$ can we find two eigenvectors $u, v$ of $A_k$ so that if we let $M = [u \ v]$ then $M$ is invertible? If there are two different eigenvalues then you can use 2(b) from assignment 2 and need not explicitly find the eigenvectors in that case.

3. Give the solutions in vector parametric form (see notes) for the plane $\pi = \{(x, y, z) : 2x - 2y + 3z = 5\}$.

4. Give the vector parametric form of all solutions to the following system of equations:

$$
\begin{align*}
2x_1 & + 4x_4 & + 6x_5 & = 14 \\
2x_1 & + 5x_4 & + 7x_5 & = 16 \\
3x_1 & + 2x_2 & + 8x_4 & + 9x_5 & = 27 \\
3x_1 & + 4x_2 & + 13x_4 & + 12x_5 & = 39
\end{align*}
$$

5. Consider the line given in vector parametric form:

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad s \in \mathbb{R}
$$

Express the set of vectors as the set of solutions to a system of equations in $x, y, z$ (two equations suffice; eliminate $s$). Then use Gaussian Elimination on this system of equations in $x, y, z$ to re-express the solutions in vector parametric form.

6. Express the solutions to

$$
\begin{bmatrix}
0 & 1 & 0 & 1 & 3 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}
= 
\begin{bmatrix}
7 \\
5 \\
0
\end{bmatrix}
$$

in vector parametric form as $x = a + sb + tc + ud$.

7. The following is helpful practice for midterm. Let

$$A = \begin{bmatrix} 5 & -6 & 1 \\ 3 & -4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Compute $\det(A - \lambda I)$. You should factor this cubic polynomial but a helpful hint is that 2 is an eigenvalue. Then for each eigenvalue find the eigenvectors. For a repeated eigenvalue, find two eigenvectors not multiples of one another.

8. Let $A$ be a $2 \times 2$ matrix with first column $x$ and second column $y$. We wish to show that $|\det(A)|$ is the area of the parallelogram formed by the two vectors $x, y$. Using assignment 2, question 2
(a), we readily deduce that \( \det(A) = 0 \) if and only if the parallelogram is degenerate with no area. So assume \( \det(A) \neq 0 \).

a) There exists an angle \( \theta \) such that \( R(\theta)\mathbf{x} = \mathbf{x}' \) and \( \mathbf{x}' \) points in the direction of the \( x \)-axis. Let \( R(\theta)\mathbf{y} = \mathbf{y}' \). Explain why the area of the parallelogram formed by \( \mathbf{x}', \mathbf{y}' \) is the same as the area of the parallelogram formed by the two vectors \( \mathbf{x}, \mathbf{y} \).

b) We can choose a value \( s \) so that if we set \( S = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \), then this shear matrix has \( S\mathbf{x}' = \mathbf{x}' \) while \( S\mathbf{y}' = \mathbf{y}'' \) where \( \mathbf{y}'' \) is parallel to the \( y \)-axis (\( x \)-coordinate 0). Explain why the area of the parallelogram formed by \( \mathbf{x}', \mathbf{y}'' \) is the same as the area of the parallelogram formed by the two vectors \( \mathbf{x}', \mathbf{y}' \).

c) Let \( B \) be a \( 2 \times 2 \) matrix with first column \( \mathbf{x}' \) and second column \( \mathbf{y}'' \). Explain why \( |\det(B)| \) is the area of the rectangular box formed by \( \mathbf{x}', \mathbf{y}'' \).

d) Using the product rule for determinants (\( \det(EF) = \det(E) \det(F) \) for any pair of \( 2 \times 2 \) matrices) and verifying that \( \det(R(\theta)) = 1 \) and \( \det(S) = 1 \), show that \( |\det(A)| \) is the area of the parallelogram formed by the two vectors \( \mathbf{x}, \mathbf{y} \).

9. (optional extra) This is harder than question 7 from assignment 2 and essentially is the reverse observation. Let \( A \) be a \( 2 \times 2 \) matrix (not necessarily diagonalizable) and define

\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\]

Assume \( \lim_{n \to \infty} \frac{x_n}{y_n} = 1 \). (This would follow if \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) is an eigenvector of \( A \) of largest eigenvalue (in absolute value)). Show that this in fact shows that \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) is an eigenvector of \( A \).

Hint: This is one possible approach. Explain why we can find \( c_1, c_2, c_3, c_4 \) with

\[
A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \quad A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\]

In effect this rewrites the matrix \( A \) with respect to the two vectors \( \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \). What do you need to show about \( c_1, c_2, c_3, c_4 \) so that \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) is an eigenvector of \( A \)?

Show that we can write \( A^n \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} \) as \( a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \end{bmatrix} \). Can you say anything about \( a, b \)?

Now compare \( A^n \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) with \( A^{n+1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = A(A^n \begin{bmatrix} 1 \\ 2 \end{bmatrix}) \).