1. Show that the $2 \times 2$ matrix of 0’s is diagonalizable (I mention this because many students confuse invertibility and diagonalizability; they are not related). Take any invertible $2 \times 2$ matrix $M$ and we see that $0M = M0$ where 0 is a diagonal matrix with 0’s on diagonal.

2. 

$$A_k = \begin{bmatrix} k & 1 \\ -1 & 3 \end{bmatrix}$$

We compute 

$$\text{det}(A_k - \lambda I) = \text{det} \begin{bmatrix} k - \lambda & 1 \\ -1 & 3 - \lambda \end{bmatrix} = \lambda^2 - (k + 3)\lambda + 3k + 1$$

Now the roots of the quadratic and hence the eigenvalues can be determined by the quadratic formula. For $(k + 3)^2 - 4(3k + 1) = k^2 - 6k + 5 = (k - 1)(k - 5)$ we see there are no eigenvalues for $(k - 1)(k - 5) < 0$ and so for $1 < k < 5$. There are two eigenvalues for $(k - 1)(k - 5) > 0$ and so for $k < 1$ or $k > 5$ there are two different eigenvalues and so the matrix is diagonalizable by 2(b) from assignment 2. There is only one eigenvalue for $(k - 1)(k - 5) = 0$ and this is for $k = 1$ and for $k = 5$.

For $k = 1$ the eigenvalue is $\lambda = 2$ and we compute eigenvectors by finding $x \neq 0$ with $(A_1 - 2I)x = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} x = 0$, namely $x = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $s \neq 0$. Thus we cannot find two eigenvectors $u, v$ so that $u \neq pv$ for any $p$.

For $k = 5$ the eigenvalue is $\lambda = 4$ and we compute eigenvectors by finding $x \neq 0$ with $(A_1 - 5I)x = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} x = 0$, namely $x = s \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for $s \neq 0$. Thus we cannot find two eigenvectors $u, v$ so that $u \neq pv$ for any $p$.

Thus $A$ is diagonalizable if and only if $k < 1$ or $k > 5$.

3. The system is already in ‘staircase’ pattern and so solution can be read off easily.

$$\{(x, y, z) : 2x - 2y + 3z = 5\} = \left\{ \begin{bmatrix} 5/2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

4. 

$$\begin{align*}
2x_1 + 4x_4 + 6x_5 &= 14 \\
2x_1 + 5x_4 + 7x_5 &= 16 \\
3x_1 + 2x_2 + 8x_4 + 9x_5 &= 27 \\
3x_1 + 4x_2 + 13x_4 + 12x_5 &= 39
\end{align*}$$

$$\begin{align*}
\begin{bmatrix} 2 & 0 & 0 & 4 & 6 & 14 \\ 2 & 0 & 0 & 5 & 7 & 16 \\ 3 & 2 & 0 & 8 & 9 & 27 \\ 3 & 4 & 0 & 13 & 12 & 39 \end{bmatrix} &\rightarrow & \begin{bmatrix} 2 & 0 & 0 & 4 & 6 & 14 \\ 2 & 0 & 0 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 & 0 & 6 \\ 0 & 4 & 0 & 7 & 3 & 18 \end{bmatrix} &\rightarrow & \begin{bmatrix} 2 & 0 & 0 & 4 & 6 & 14 \\ 0 & 2 & 0 & 2 & 0 & 6 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} &\rightarrow & \begin{bmatrix} 1 & 0 & 0 & 2 & 3 & 7 \\ 0 & 1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} &\rightarrow & \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{align*}$$
for which the solutions are
\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
\end{bmatrix} =
\begin{bmatrix}
  3 \\
  1 \\
  0 \\
  2 \\
  0 \\
\end{bmatrix} + s
\begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0 \\
  1 \\
\end{bmatrix} + t
\begin{bmatrix}
  -1 \\
  1 \\
  0 \\
  -1 \\
  0 \\
\end{bmatrix}
\]

5. The line
\[
\begin{bmatrix}
  x \\
  y \\
  z \\
\end{bmatrix} =
\begin{bmatrix}
  1 \\
  1 \\
  1 \\
\end{bmatrix} + s
\begin{bmatrix}
  2 \\
  -1 \\
  0 \\
\end{bmatrix} \text{ yields equations } y = 1 + 2s \quad \text{and } z = 1 - s
\]
This yields three expressions for s: \( s = x - 1, s = 1/2y - 1/2, s = 1 - z \). Equating them (first and second, then first and third will work) yield equations in \( x, y, z \) with no \( s \), namely: \( x - 1/2y = 1/2 \) and \( x + z = 2 \). We solve
\[
\begin{bmatrix}
  1 & -1/2 & 0 & : & 1/2 \\
  1 & 0 & 1 & : & 2 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
  1 & -1/2 & 0 & : & 1/2 \\
  0 & 1/2 & 1 & : & 3/2 \\
\end{bmatrix}
\]
Now the vector parametric form of the solutions is
\[
\begin{bmatrix}
  x \\
  y \\
  z \\
\end{bmatrix} =
\begin{bmatrix}
  1/2 \\
  3 \\
  0 \\
\end{bmatrix} + s
\begin{bmatrix}
  -1 \\
  -2 \\
  1 \\
\end{bmatrix}.
\]
This is seen to yield the same line we started with, not surprisingly.

6. Express the solutions to
\[
\begin{bmatrix}
  0 & 1 & 0 & 1 & 3 \\
  0 & 0 & 0 & 1 & 2 \\
  0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
\end{bmatrix} =
\begin{bmatrix}
  7 \\
  5 \\
  0 \\
\end{bmatrix}
\]
in vector parametric form as \( \mathbf{x} = \mathbf{a} + s\mathbf{b} + t\mathbf{c} + u\mathbf{d} \).

The set of solutions is
\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  2 \\
  0 \\
  5 \\
  0 \\
\end{bmatrix} + s
\begin{bmatrix}
  1 \\
  0 \\
  0 \\
  0 \\
  0 \\
\end{bmatrix} + t
\begin{bmatrix}
  0 \\
  0 \\
  1 \\
  0 \\
  0 \\
\end{bmatrix} + u
\begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0 \\
  1 \\
\end{bmatrix} : s, t, u \in \mathbb{R}
\]

7. The following is helpful practice for midterm. Let
\[
A = \begin{bmatrix}
  5 & -6 & 1 \\
  3 & -4 & 1 \\
  0 & 0 & 2 \\
\end{bmatrix}
\]
\[
\det(A - \lambda I) = \det\left( \begin{bmatrix}
  5 - \lambda & -6 & 1 \\
  3 & -4 - \lambda & 1 \\
  0 & 0 & 2 - \lambda \\
\end{bmatrix} \right) = (5 - \lambda)((-4 - \lambda)(2 - \lambda)) + 6(3(2 - \lambda)) + 1 \\
= (2 - \lambda)((-4 - \lambda)(2 - \lambda) + 18) = (2 - \lambda)(\lambda^2 - \lambda - 2) = (2 - \lambda)(\lambda - 2)(\lambda + 1)
\]
The cubic is factored into 3 linear factors where 2 is an eigenvalue that is a repeated root and -1 is also an eigenvalue. We solve using Gaussian Elimination.

\[ \lambda = 2 \text{ Solve } (A - 2I)v = 0 \]

\[
\begin{bmatrix}
3 & -6 & 1 \\
3 & -6 & 1 \\
0 & 0 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
3 & -6 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

We read off solutions

\[
\left\{ s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}
\]

\[ \lambda = -1 \text{ Solve } (A + I)v = 0 \]

\[
\begin{bmatrix}
6 & -6 & 1 \\
3 & -3 & 1 \\
0 & 0 & 3
\end{bmatrix} \rightarrow \begin{bmatrix}
6 & -6 & 1 \\
0 & 0 & 1/2 \\
0 & 0 & 0
\end{bmatrix}
\]

We read off solutions

\[
\left\{ s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\}
\]

8. Let \( A \) be a \( 2 \times 2 \) matrix with first column \( \mathbf{x} \) and second column \( \mathbf{y} \). We wish to show that \( |\det(A)| \) is the area of the parallelogram formed by the two vectors \( \mathbf{x}, \mathbf{y} \). Using assignment 2, question 2 (a), we readily deduce that \( \det(A) = 0 \) if and only if the parallelogram is degenerate with no area. So assume \( \det(A) \neq 0 \).

a) It is clear geometrically that we can rotate by some angle \( \theta \) to take \( \mathbf{x} \) to \( \mathbf{x}' \) where \( \mathbf{x}' \) points in the direction of \( x \)-axis. Thus \( R(\theta)\mathbf{x} = \mathbf{x}' \). Let \( \mathbf{y}' = R(\theta)\mathbf{y} \). Now the parallelogram formed by \( \mathbf{x}', \mathbf{y}' \) is the same as the parallelogram formed by the two vectors \( \mathbf{x}, \mathbf{y} \) except rotated so it has the same area.

b) Consider the shear matrix \( S = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \). We have \( S\mathbf{x}' = \mathbf{x}' \) since \( \mathbf{x}' \) is a multiple of \( [10] \). Let \( \mathbf{y}' = \begin{bmatrix} p \\ q \end{bmatrix} \). It is now straightforward to solve for \( s \) so that \( p + sq = 0 \) and so if we let \( \mathbf{y}'' = S\mathbf{y}' \) then has \( \mathbf{y}'' \) points in the direction of the \( y \)-axis \( (q > 0) \) or its opposite \( (q < 0) \). The operation of shearing preserves the base of the parallelogram \( \mathbf{x}' \) while maintaining the height and so the area of the parallelogram formed by \( \mathbf{x}', \mathbf{y}'' \) is the same as the area of the parallelogram formed by the two vectors \( \mathbf{x}', \mathbf{y}' \).

c) The area of the rectangular box formed by \( \mathbf{x}', \mathbf{y}'' \) is the base times the height. Assume \( \mathbf{x}' = \begin{bmatrix} a \\ 0 \end{bmatrix} \) and \( \mathbf{y}'' = \begin{bmatrix} 0 \\ b \end{bmatrix} \). Then the area of the rectangular box formed by \( \mathbf{x}', \mathbf{y}'' \) is \( |ab| \) which is the same as \( |\det(B)| \) where \( B \) is the matrix whose first column is \( \mathbf{x}' \) and whose second column is \( \mathbf{y}'' \). Note that we must use the absolute value because we do not know the sign of \( b \), while we can choose \( \theta \) so that \( a > 0 \).

d) Now \( SR(\theta)A = B \) and so \( \det(B) = \det(S)\det(R(\theta))\det(A) \). Now \( \det(R(\theta)) = 1 \) and \( \det(S) = 1 \) and so \( \det(A) = \det(B) \). But by a), b), c), \( |\det(B)| \) is the area of the parallelogram formed by the two vectors \( \mathbf{x}, \mathbf{y} \) and so this is also true for \( A \).
9. (optional extra) This is harder than question 7 from assignment 2 and essentially is the reverse observation. Let \( A \) be a \( 2 \times 2 \) matrix (not necessarily diagonalizable) and define

\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\]

Assume \( \lim_{n \to \infty} \frac{x_n}{y_n} = 1 \). (This would follow if \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) is an eigenvector of \( A \) of largest eigenvalue (in absolute value)). Show that this in fact shows that \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) is an eigenvector of \( A \).

My hint is just one way to make this work. We note that \( \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) is a basis for \( \mathbb{R}^2 \). namely any vector in \( \mathbb{R}^2 \) can be expressed as a linear combination of these two vectors. They form a new coordinate system much as in our white/blue example. We can express

\[
A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\]

This is essentially writing the linear transformation \( f(x) = Ax \) with input and output written with respect to this basis.

We are done if we can show \( c_2 = 0 \) which makes \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) an eigenvector of \( A \) of eigenvalue \( c_1 \). Now assume

\[
A^n \begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_5(n) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_6(n) \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

where I use \( c_5(n), c_6(n) \) since the coefficients would vary with the choice of \( n \). Thus \( x_n = c_5(n) + c_6(n) \) and \( y_n = c_5(n) + 2c_6(n) \) and

\[
\lim_{n \to \infty} \frac{x_n}{y_n} = \lim_{n \to \infty} \frac{c_5(n) + c_6(n)}{c_5(n) + 2c_6(n)} = 1.
\]

I think that the key idea in making a clear proof is expressing these coefficients as a function of \( n \). Dividing top and bottom by \( c_5(n) \) (\( c_5(n) \neq 0 \) for large \( n \) else the limit will not work) and we derive \( \lim_{n \to \infty} \frac{c_6(n)}{c_5(n)} = 0 \). Thus \( c_6(n) \) is small in magnitude compared to \( c_5(n) \) which may seem obvious but I’ve given you a proof.

We need to show that \( c_2 = 0 \) in order to show that \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) is an eigenvector of \( A \). We compute

\[
A^{n+1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = A(c_5(n) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_6(n) \begin{bmatrix} 1 \\ 2 \end{bmatrix}) = (c_5(n)c_1 + c_6(n)c_3) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (c_5(n)c_2 + c_6(n)c_4) \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

Thus

\[
\frac{x_{n+1}}{y_{n+1}} = \frac{c_5(n)c_1 + c_6(n)c_3 + c_5(n)c_2 + c_6(n)c_4}{c_5(n)c_1 + c_6(n)c_3 + 2c_5(n)c_2 + 2c_6(n)c_4} = \frac{c_1 + \frac{c_6(n)}{c_5(n)}c_2 + \frac{c_6(n)}{c_5(n)}c_4}{c_1 + \frac{c_6(n)}{c_5(n)}c_3 + 2c_2 + 2\frac{c_6(n)}{c_5(n)}c_4}
\]

the last equation obtained by dividing by \( c_5(n) \). Thus

\[
\lim_{n \to \infty} \frac{x_{n+1}}{y_{n+1}} = \frac{c_1 + c_2}{c_1 + 2c_2} = 1.
\]

We deduce that \( c_2 \) equals 0.

There are other approaches using limits etc. I like the idea that either \( c_2 = 0 \) or it messes up the limit as \( n \to \infty \) but we see that by simply multiplying by one further \( A \). You can imagine a generalization. And note that it gives the hint that when seeing some behaviour in the limit it might be a (dominant) eigenvector.