1. Compute

\[ \text{i) } \det \begin{bmatrix} x & 0 & 0 \\ 10 & x & 0 \\ 52 & 223 & 1 \end{bmatrix} = x^2 \text{ (triangular matrix and } \det(A) = \det(A^T)) \]

\[ \text{ii) } \det \begin{bmatrix} 99 & 100 & 101 \\ 0 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} = 0 \text{ (row of 0's)} \]

\[ \text{iii) } \det \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = 5 \text{ (expansion about first row will work or gaussian elimination but there are more clever ways.)} \]

2. (nice problem) Consider three 3 digit numbers write in decimal digits as \(abc\), \(def\) and \(ghi\). (e.g. \(abc = a \cdot 10^2 + b \cdot 10 + c\)). Assume that each of the numbers \(abc\), \(def\) and \(ghi\) is divisible by 17. Show that

\[ \det \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \]

is also divisible by 17.

Let

\[ A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \]

What we do is apply row operations adding 100 times the first row to the third row and 10 times the second row to the third row and we obtain

\[ B = \begin{bmatrix} a & d & g \\ b & e & h \\ abc & def & ghi \end{bmatrix} \]

where \(\det(B) = \det(A)\). We may assume \(abc = 17j\), \(def = 17k\) and \(ghi = 17\ell\) for integers \(j, k, \ell\) so that

\[ B = \begin{bmatrix} a & d & g \\ b & e & h \\ 17j & 17k & 17\ell \end{bmatrix} \]

Now

\[ \det(B) = 17 \det \begin{bmatrix} a & d & g \\ b & e & h \\ j & k & \ell \end{bmatrix}. \]

What multiplies 17 is the determinant of an integer matrix and so we conclude 17 divides \(\det(A)\).

3. \[ A = \begin{bmatrix} x & 1 & 1 \\ 1 & x & 1 \\ x & 2 & 1 \end{bmatrix} \]

\[ \det(x(x - 2) - (1 - x) + (2 - x^2) = 1 - x \]
Thus we can determine the inverse when \( 1 - x \neq 0 \).

\[
A^{-1} = \begin{bmatrix}
\frac{x-2}{1-x} & \frac{1}{1-x} & 1 \\
-1 & 0 & 1 \\
\frac{2-x^2}{1-x} & \frac{-x}{1-x} & -1 - x
\end{bmatrix}
\]

You can check this. I did.

4. Assume that \( A = MBM^{-1} \). We can show that \( \det(A - \lambda I) = \det(B - \lambda I) \) as follows. \( \det(A - \lambda I) = \det(MBM^{-1} - \lambda I) = \det(MBM^{-1} - M(\lambda I)M^{-1}) = \det(M(B - \lambda I)M^{-1}) = \det(M) \det(B - \lambda I) \det(M^{-1}) = \det(B - \lambda I) \). We are using the product rule at several points including our cancellation that \( \det(M) \det(M^{-1}) = 1 \).

5. Let \( A \) be an \( n \times n \) matrix with an eigenvector \( v \) of eigenvalue \( t \). Assume we can obtain an invertible matrix \( M \) which has \( v \) as its first column. We are given \( Av = tv \) and compute \( AM = [tv \cdots] \). We can write \( [tv \cdots] = [tv B] \) where \( B \) is an \( n \times (n - 1) \) matrix. I claim

\[
[tv B] = M \begin{bmatrix}
t \\
0 \\
0 \\
\vdots
\end{bmatrix}
\]

and thus

\[
M^{-1}AM = \begin{bmatrix}
t \\
0 \\
0 \\
\vdots
\end{bmatrix}
\]

and so the first column is a \( t \) followed by 0’s.

Using expansion about the first column, we can easily compute \( \det(M^{-1}AM - \lambda I) = (t - \lambda)p(\lambda) \) where \( p(\lambda) \) is a polynomial in \( \lambda \) of degree \( n - 1 \). This means \( A \) has \( t \) as an eigenvalue.

6. (from a test) Assume \( A \) is a \( 3 \times 3 \) matrix, and \( M \) is an invertible matrix with \( A = MDM^{-1} \), where \( D \) is the diagonal matrix

\[
D = \begin{bmatrix}
2 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 4
\end{bmatrix}
\]

Show that \( (A - 2I)(A - 3I)(A - 4I) = 0 \) where 0 denotes the \( 3 \times 3 \) matrix of 0’s.

We note that

\[
\]

\[
= M(D - 2I)M^{-1}M(D - 3I)M^{-1}M(D - 4I)M^{-1} = M(D - 2I)(D - 3I)(D - 4I)M^{-1}
\]

\[
= M \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{bmatrix} \begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
-2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{bmatrix} M^{-1}
\]

\[
= M0M^{-1} = 0.
\]

7. Show that the \( 4 \times 4 \) matrix of 1’s is diagonalizable. Try to generalize to the \( n \times n \) matrix of 1’s.
I’ll go directly to the $n \times n$ case. You might note that 0 is an eigenvalue and then (with very easy Gaussian Elimination) find the eigenvectors

$$
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
-1 & 0 & \cdots & 0 \\
0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -1
\end{bmatrix}
$$

We also might guess the eigenvector of 1’s which you can verify has eigenvalue $n$. Now we check

$$
M = \begin{bmatrix}
1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
-1 & 0 & 0 & 0 & \cdots & 0 & 1 \\
0 & -1 & 0 & 0 & \cdots & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -1 & 1
\end{bmatrix}
$$

8. Let $A = (a_{ij})$ be an $n \times n$ matrix with integral entries such that the diagonal entries are all not divisible by 3 ($a_{ii}$ is not evenly divisible by 3) and all off diagonal entries are divisible by 3 ($a_{ij}$ is divisible by 3 for $i \neq j$). Show that $A$ has $\det(A) \neq 0$, i.e. $A$ is invertible.

I have seen a variety of proofs of this result. Note if we multiply an integer by a number divisible by 3 then the product is divisible by 3. Also if we multiply two numbers not divisible by 3, then the product is not divisible by 3. Also 0 is divisible by 3.

Use induction on $n$, namely we use $H(k)$ to denote the statement that The determinant of a $k \times k$ matrix with integral entries such that the diagonal entries are all not divisible by 3 ($a_{ii}$ is not evenly divisible by 3) and all off diagonal entries are divisible by 3 ($a_{ij}$ is divisible by 3 for $i \neq j$) is itself not divisible by 3. This is a stronger induction hypothesis than you might initially choose, making this question harder. It is automatic that if $\det(A)$ is not divisible by 3 (and is an integer), then $\det(A) \neq 0$. But the reverse is not true.

We note that $H(1)$ is trivially true. We could verify that $H(2)$ is true by noting

$$
\det \begin{bmatrix}
a & 3b \\
3c & d
\end{bmatrix} = ad - 9bc.
$$

If we assume $a, d$ are not divisible by 3 then neither is $ad$ and so neither is $ad - 9bc$. For some $k \geq 1$, assume $H(k)$ is true. We will show $H(k+1)$ is true. Let $A$ be the $(k + 1) \times (k + 1)$ matrix with

$$a_{ij} = \begin{cases}
\text{not divisible by 3} & \text{if } i = j \\
\text{divisible by 3} & \text{if } i \neq j
\end{cases}
$$

Now we use the recursive formula for $\det A$:

$$
\det(A) = (-1)^{1+1}a_{11} \det M_{11} + (-1)^{1+2}a_{12} \det M_{12} + \cdots + (-1)^{1+k+1}a_{1(k+1)} \det M_{1(k+1)}.
$$

We note that $a_{12}, a_{13}, \ldots, a_{1(k+1)}$ are all divisible by 3 and so $(-1)^{1+j}a_{1j} \det M_{1j}$ is divisible by 3 for $j \neq 1$. Also, $H(k)$ implies that $\det M_{11}$ is not divisible by 3, $a_{11}$ is not divisible by 3 and so $(-1)^{1+1}a_{11} \det M_{11}$ is not divisible by 3. We have that $\det(A)$ is the sum of a number not divisible by 3 for $j \neq 1$. Hence $H(k+1)$ is true.

Now given that $H(1)$ is true and $H(k)$ implies $H(k+1)$ for all $k$, then by the principle of mathematical induction, $H(k)$ is true for all $k$ and hence $H(n)$ is true. Thus $H(n)$ is true and so the $\det(A)$ is not divisible by 3 and so $\det(A) \neq 0$. 