1. Let $U, W$ be subspaces of a vector space $V$. Show that $U \cap W = \{ \mathbf{v} : \mathbf{v} \in U \text{ and } \mathbf{v} \in W \}$ is a subspace of $V$. Comment on whether $U \cup W$ is always a vector space.

2. Let $\{u_1, u_2, u_3\}$ be a basis for a vector space $V$. Then if we define $\mathbf{v}_1 = u_1 + 2u_3$, $\mathbf{v}_2 = u_1 + 2u_2 + 3u_3$, $\mathbf{v}_3 = u_2 - u_3$, show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a basis for $V$.

3. (from a test)

   Let $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$, $\mathbf{w}_3 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$

   NOTE: $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 4 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$

   Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation satisfying

   $$f(\mathbf{w}_1) = \mathbf{w}_2 - \mathbf{w}_3, \quad f(\mathbf{w}_2) = -\mathbf{w}_2 + \mathbf{w}_3, \quad f(\mathbf{w}_3) = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3.$$  

   a) Give the matrix representation of $f$ with respect to the basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

   b) Give the matrix representation of $f$ where the input $\mathbf{x}$, is written with respect to the basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ and the output $f(\mathbf{x})$ is written with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (the standard basis).

   c) Is $\mathbf{w}_1$ in the range of $f$?

4. (from a test)

   Let $\mathbf{z}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{z}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{z}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

   NOTE: $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

   Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation satisfying

   $$T(\mathbf{z}_1) = 2\mathbf{z}_2, \quad T(\mathbf{z}_2) = 2\mathbf{z}_2, \quad T(\mathbf{z}_3) = \mathbf{z}_1 + \mathbf{z}_2.$$  

   a) Give the matrix representation of $T$ with respect to the basis $\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$.

   b) Give the matrix representation of $T$ with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (the standard basis).

   c) Give the explicit matrix with integer entries.

   c) Give the matrix representing $T^2$ with respect to the basis $\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$. What is the rank of the matrix representing $T^2$ with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$?

5. (from a test) Solve the system of differential equations

   $$\frac{dx_1}{dt} = -5x_1(t) + 6x_2(t), \quad \frac{dx_2}{dt} = -x_1(t)$$

   First find the general solution for $x_1(t), x_2(t)$ as a function of $x_1(0), x_2(0)$. Given $x_1(0) = 6$ and $x_2(0) = 2$, find the solutions explicitly and compute $\lim_{t \to \infty} \frac{x_1(t)}{x_2(t)}$. 

MATH 223 Assignment #7 due Friday Nov 5. Richard Anstee
6. Consider a $10 \times 10$ diagonalizable matrix $A$ with $\text{tr}(A) = 0$ and such that $A^2 + A = J + 2I$ where $J$ is the $10 \times 10$ matrix of 1’s. Assume $A$ has eigenvalue 3 with multiplicity 1. Following assignment 5, question 7, we know that $J + 2I$ has eigenvalue 12 with multiplicity 1 and eigenvalue 2 with multiplicity 9. Verify that $A$ has eigenvalue 3 with multiplicity 1, eigenvalue 1 with multiplicity 5 and eigenvalue -2 with multiplicity 4. You might wish to consider assignment 5, question 6.

7. Let $U$ and $V$ be two 5-dimensional subspaces of $\mathbb{R}^9$. Show that there is a nonzero vector in $U \cap V$, the intersection of $U$ and $V$. Using Gaussian elimination should make this reasonable. You may find it helpful to consider the case of two 2-dimensional subspaces of $\mathbb{R}^3$ first but note that you won’t be able to use ideas of lines and planes in $\mathbb{R}^3$.

8. (optional) The following question is from a Putnam exam and perhaps I’ve given enough ideas to make it doable.

a) Show that it is impossible to have vectors $u, v, w, t$ with

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = uv^T + wt^T$$

b) We can encode polynomials in two variables $x, y$ as a matrix so that the polynomial $3x^2y - xy^2 + 5x^3y^3$ is encoded by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Thus the $i, j$ entry corresponds to the coefficient of $x^{i-1}y^{j-1}$. Say we have a polynomial $p(x) = x + 2x^2$ and polynomial $q(y) = 5 - y^2$ then we have $p(x)q(y)$ encoded as

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & -1 & 0 \\ 10 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & -1 & 0 \end{bmatrix}.$$

Show that in general that a product of two polynomials $p(x), q(y)$ can be encoded as $uv^T$ where $u$ encodes $p(x)$ and $v$ encodes $q(y)$. Can there exist polynomials $p(x), q(y), r(x), s(y)$ (each of maximum degree 3) such that $p(x)q(y) + r(x)s(y) = 1 + xy + x^2y^2$?