Math 223

Assignment 9

Due Wednesday Dec 1 in class.

1. Find an orthonormal basis for \( \mathbb{R}^3 \) by applying Gram-Schmidt to the three vectors:

\[
\begin{bmatrix}
1 \\
2 \\
-2
\end{bmatrix}, \quad \begin{bmatrix}
4 \\
3 \\
2
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix}
\]

Please recall that we can change a vector with fractional entries to one with integral entries by rescaling and we do not need to normalize until the last step.

Question 2 is likely (very likely!) to be similar to a question on the final. The other questions are all related to exam questions. Please note the very important fact that an \( n \times n \) symmetric matrix always is diagonalizable with an orthonormal basis of eigenvectors for \( \mathbb{R}^n \). This will be covered in class but perhaps not completed before the due date of this assignment. You can see some practice questions involving symmetric matrices in the sample final exams.

2. Find orthonormal bases of eigenvectors for the following matrices (you can find such questions on every exam):

\[
A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 10 & 10 \\ 10 & 5 & 0 \\ 10 & 0 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & -2 & 1 \\ -2 & 2 & -2 \\ 1 & -2 & -1 \end{bmatrix}
\]

Hint: for \( B \), 0 is an eigenvalue, and for \( C \), -2 is an eigenvalue.

3. (adapted from an exam) Let \( A \) be a \( 3 \times 3 \) matrix with \( \det(A - \lambda I) = -(\lambda^3 + a\lambda^2 + b\lambda + c) \). Show that if \( A \) is diagonalizable, then the following equation is true

\[
A^3 + aA^2 + bA + cI = 0
\]

(this equation is in fact true for any \( 3 \times 3 \) matrix and is a special case of the Cayley-Hamilton Theorem).

4. Let \( \{u_1, u_2, \ldots, u_k\} \) be non-zero vectors satsifying \( u_i \cdot u_j = 0 \) for all pairs \( i, j \) with \( i \neq j \). Show that \( \{u_1, u_2, \ldots, u_k\} \) are linearly independent.

5. (from an exam) Let \( A \) be an \( n \times n \) symmetric matrix with eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) (some may repeat) and an orthonormal basis of eigenvectors \( u_1, u_2, \ldots, u_n \) (\( A u_i = \lambda_i u_i \)). Then show that

\[
A = \sum_{i=1}^{n} \lambda_i u_i u_i^T.
\]

(thus \( A \) is a sum of \( n \) symmetric rank 1 matrices)

6. (from an exam) Consider a matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). We could attempt to solve for \( A^{-1} \) by letting

\[
A^{-1} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}
\]

with four variables \( x, y, z, t \) and then \( AA^{-1} = I \) becomes a system of 4 equations in 4 unknowns with an associated \( 4 \times 4 \) matrix \( B \). What is the rank of the \( 4 \times 4 \) system of equations assuming \( A^{-1} \) exists? Explain. Can you say anything about the rank of \( B \) if \( \det(A) = 0 \)? Explain.

7. (from an exam) You are attempting to solve for \( x_1, x_2, x_3 \) in the matrix equation \( Ax = b \) where

\[
A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}
\]
Find a ‘least squares’ choice \( \hat{b} \) in the column space of \( A \) (and hence with \( \|b - \hat{b}\|^2 \) being minimized) and then solve the new system \( Ax = \hat{b} \) for \( x_1, x_2, x_3 \). You can check your choice of \( \hat{b} \) by testing if \( b - \hat{b} \) is orthogonal to the column space of \( A \).

8. When considering a permutation \( \sigma \) of \( n \) symbols \( \{1, 2, \ldots, n\} \), there is many ways to represent \( \sigma \) as a product of transpositions/interchanges of two symbols. There is an invariant \( sgn(\sigma) \) which is +1 if the number of transpositions to represent \( \sigma \) is always odd and is 0 if number of transpositions is always even. Show that this function \( sgn(\sigma) \) is well defined for all \( \sigma \).

Think of \( \sigma \) as the \( n \times n \) matrix \( P_\sigma \) with \( p_{ij} = 1 \) if \( \sigma(j) = i \) and 0 otherwise. There is one 1 in each row and one 1 in each column. The idea is to use the determinant function. Can you apply row exchanges to take \( P_\sigma \) to \( I \)? Consider \( \det(P_\sigma) \).

It is true that one flip this idea and define the determinant function using \( sgn(\sigma) \) while proving that \( sgn(\sigma) \) is well defined by a separate argument.

9. (optional, harder) You are given the following vector recurrence for \( n \geq 1 \).
\[
x_n = b + Ax_{n-1}
\]
where \( x_0 \) has been given to you to begin the recurrence. We are given that \( A \) is a \( 3 \times 3 \) matrix with eigenvalues
\[
|\lambda_3| < |\lambda_2| < \lambda_1 = 1,
\]
and associated eigenvectors \( v_3, v_2, v_1 \). Assume \( x_0 = x_1v_1 + x_2v_2 + x_3v_3 \) and \( b = b_1v_1 + b_2v_2 + b_3v_3 \) with \( x_1 \neq 0 \) and \( b_1 \neq 0 \). Show that
\[
\lim_{n \to \infty} x_n - x_{n-1} = b_1v_1.
\]
Idea: expand \( x_n \) and \( x_{n-1} \) down to \( x_0 \) and then subtract them.
Comment: I was using this in a stochastic dynamic programming problem in MATH 441.

10. (optional, harder) Let \( C \) be a \( 2 \times 3 \) matrix and let \( D \) be a \( 3 \times 2 \) matrix. Assume
\[
DC = \begin{bmatrix}
-2 & -1 & 2 \\
0 & 0 & 3 \\
0 & 0 & 5
\end{bmatrix}
\]
Compute \( \det(CD) \).

Hint: We first note that \( DC \) is a rank 2 matrix which means that \( \text{rank}(C) \geq 2 \) and \( \text{rank}(D) \geq 2 \) and so \( \text{rank}(C) = \text{rank}(D) = 2 \). In particular, the row space of \( DC \) must be contained in the row space of \( C \) and so
\[
C = S \begin{bmatrix}
-2 & -1 & 2 \\
0 & 0 & 1
\end{bmatrix},
\]
where \( S \) is some invertible \( 2 \times 2 \) matrix. Other choices are possible. Now the column space of \( DC \) must be contained in the column space of \( D \) and so
\[
D = \begin{bmatrix}
1 & 0 \\
0 & 3 \\
0 & 5
\end{bmatrix} T,
\]
where \( T \) is some invertible \( 2 \times 2 \) matrix. Show that \( TS = I \) in order for \( DC \) to satisfy the matrix equation. Then write \( CD \) and compute \( \det(CD) \).