Field Axioms

A field is a set of elements $\mathbf{F}$ which we call *scalars* when used in a vector space. A quick check verifies that the real numbers $\mathbf{R}$, the complex numbers $\mathbf{C}$ and the rational numbers $\mathbf{Q}$ all are examples of fields. The integers $\mathbf{Z}$ are not (because no multiplicative inverses). There are other useful examples of fields which we do not use in this course.

In what follows α, β, γ are arbitrary elements of $\mathbf{F}$

$\forall \alpha, \beta \in \mathbf{F},\ \alpha + \beta = \beta + \alpha$ (commutativity)

$\forall \alpha, \beta, \gamma \in \mathbf{F},\ \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ (associativity)

$\exists 0$ with $\forall \alpha \in \mathbf{F},\ \alpha + 0 = \alpha$

$\forall \alpha \in F, \exists - \alpha \in F$ with $\alpha + (-\alpha) = 0$

$\forall \alpha, \beta \in \mathbf{F},\ \alpha\beta = \beta\alpha$ (commutativity)

$\forall \alpha, \beta, \gamma \in \mathbf{F},\ \alpha(\beta\gamma) = (\alpha\beta)\gamma$ (associativity)

$\exists 1$ with $\forall \alpha \in \mathbf{F},\ \alpha 1 = \alpha$

$\forall \alpha \in \mathbf{F}$ with $\alpha \neq 0, \exists \alpha^{-1}$ with $\alpha\alpha^{-1} = 1$

$\forall \alpha, \beta, \gamma \in \mathbf{F},\ \alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

Vector Spaces Axioms

We have as set V of *vectors* and a field $\mathbf{F}$ (typically $\mathbf{R}$ in this course) of scalars.

In what follows $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are arbitrary elements of V and k, ℓ are arbitrary elements of $\mathbf{F}$.

$\forall \mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} \in V$ (closure)

$\forall \mathbf{u} \in V \;\; \forall k \in \mathbf{F}$, $k\mathbf{u} \in V$ (closure)

$\forall \mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity)

$\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associativity)

$\exists \mathbf{0}$ (that is unique) with $\forall \mathbf{u} \in V$, $\mathbf{0} + \mathbf{v} = \mathbf{v}$

$\forall k \in \mathbf{F} \; \forall \mathbf{u}, \mathbf{v} \in V \;\; k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$ (distributivity)

$\forall k, \ell \in \mathbf{F} \; \forall \mathbf{u} \in V \;\; (k + \ell)\mathbf{u} = k\mathbf{u} + \ell\mathbf{u}$ (distributivity)

$\forall k, \ell \in \mathbf{F} \; \forall \mathbf{u} \in V \;\; k\ell\mathbf{u} = k(\ell\mathbf{u})$

$\forall \mathbf{u} \in V \;\; 1 \cdot \mathbf{u} = \mathbf{u}$