MATH 223: Fischer's Inequality.

This example shows how the Rank or dimension idea can have valuable and unexpected consequences. There is no 'direct' proof known of this result.

We must first define *block designs* whose original motivations were from Statistics. The letters BIBD refer to Balanced Incomplete Block Design. A (b, v, k, r, λ)-BIBD is a family $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ of b subsets of $\{1, 2, \ldots, v\}$. We require $0 < \lambda$ and $k < v - 1$. We can form a $v \times b$ matrix $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } i \in B_j \\ 0 & \text{if } i \notin B_j \end{cases}.$$

Thus row i give the blocks containing element i and column j gives the elements in block B_j

The special properties of the subsets are the following where the third is the most important.

$$|B_j| = k \text{ for } j = 1, 2, \ldots, b.$$

$$|\{j : i \in B_j\}| = r \text{ for } i = 1, 2, \ldots, v.$$

For each pair $\{i, p\} \subseteq \{1, 2, \ldots, v\}$ there are exactly λ blocks j with $\{i, p\} \subseteq B_j$.

We wish to establish Fischer's Inequality:

$$v \leq b.$$

The special properties of our matrix A give us the matrix equation

$$AA^T = (r - \lambda)I + \lambda J.$$

To verify this equation you might note that the 1's in each row of A corresponds to the blocks containing that element. Now an entry of AA^T is dot product of a row of A and a column of A^T but we also know that a column of A^T is a row of A. Now the inner product of two such rows is either r if the rows are the same or λ if the rows are different by our last mentioned property.

Now, by an exercise, we find that

$$\det((r - \lambda)I + \lambda J) = (r + (v - 1)\lambda)(r - \lambda)^{v-1}.$$

We note that $r > \lambda$ follows from $k < v - 1$ and the equality $\frac{vr(k-1)}{2} = \frac{\lambda v(v-1)}{2}$ and thus the matrix is invertible. Hence $(r - \lambda)I + \lambda J$ is of rank v. Hence $\text{rank}(AA^T) = v$ and so $\text{rank}(A) \geq v$ (column space of AA^T is contained in column space of A). Given that A is a $v \times b$ matrix, we already have that $\text{rank}(A) \leq b$ and so we deduce the inequality $v \leq b$.

One famous block design, a (7,7,3,3,1)-BIBD known as the Fano Plane, has the following matrix to represent it:

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

You may check that $AA^T = 2I + J$. Also $A^T A = AA^T$!