MATH 223: Moore graphs.

A *graph* $G = (V, E)$ consist of a finite set V of vertices (singular is vertex) and a set E of edges, each edge consisting of an unordered pair of vertices. We define the adjacency matrix A :

$$A = (a_{ij}), \qquad a_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \in E \\ 0 & \text{if } \{i,j\} \notin E \end{cases}$$

Such a (0,1)-matrix would be symmetric and we can use our theorems about symmetric matrices.

In a graph, we can define the distance $d_G(x, y)$ between vertex x and vertex y as the length, in number of edges, of the shortest path between x and y. Then a graph has diameter:

$$\text{diam}(G) = \max_{x,y \in V} d_G(x, y)$$

We are sometimes interested in graphs of small diameter which have many vertices with respect to the number of edges. This approach has been used in designing computer clusters.

For the rest of this discussion, we will assume that $\text{diam}(G) = 2$. Let the degree of a vertex be the number of edges incident with that vertex. We say a graph is regular of degree k if every vertex has degree k. Now a graph of degree at most k and with diameter 2 has at most $1 + k + k(k-1)$ vertices (we deduce the bound by noting that for a given vertx v, there are at most k vertices at distance 1 and at most $k(k-1)$ vertices at distance 2 and no vertices at greater distance from v).

Assume that we have a graph which is regular of degree k and of diameter 2 and with $1+k+k(k-1)$ vertices. Such a graph is called a *Moore Graph*. We deduce that the graph has no triangles or four cycles and the adjacency matrix satisfies the interesting matrix equation where J is the matrix of 1's.

$$A^2 + A = (k-1)I + J.$$

This follows since the i, j-entry of A is 1 if and only if there is an edge $\{i, j\}$. The i, j-entry of A^2 is nonzero if and only if there is a vertex k with edges $\{i, k\}$, $\{k, j\}$ in E. If there is a vertex k with edges $\{i, k\}$, $\{k, j\}$ in E, then because there are no triangles, there will be no edge $\{i, j\}$ and because there are no four cycles, there is only one possible choice for k. Note that if there is an edge $\{i, j\}$, then no such k can exist.

Now we note that the eigenvalues for $(k-1)I + J$ are $k + n - 1$ with multiplicity 1 and $k-1$ with multiplicity $n-1$. We use the information about the eigenvalues for $A^2 + A$ to deduce information about the eigenvalues for A. because A is symmetric, we know that A is diagonalizable with $AM = MD$ where M is invertible and D is a diagonal matrix of the eigenvalues. We note that the trace of A is zero and hence the sum of the eigenvalues, counted with multiplicities, is zero.

Now, if $AM = MD$, then $(A^2 + A)M = M(D^2 + D)$. We deduce that there are $n-1$ eigenvalues for A satisfying $\lambda^2 + \lambda = k - 1$ and there is one eigenvalue for A satisfying $\lambda^2 + \lambda = k + (n-1)$. For this latter case, we note that k is an eigenvalue for A (the eigenvector is $\mathbf{1}$), and $k^2 + k = k + (n-1)$ using $n = 1 + k + k(k-1)$. So now let

$$s \text{ eigenvalues } \frac{-1+\sqrt{4k-3}}{2}, \qquad (n-1-s) \text{ eigenvalues } \frac{-1-\sqrt{4k-3}}{2}$$

recall that the sum of the eigenvalues of A is $\text{tr}(A) = 0$.

If $4k-3$ is not a square of an integer, then we must have $s = (n-1)/2$ so that the irrational parts involving $\sqrt{4k-3}$ cancel. But then the sum of the eigenvalues is $k + (n-1)/2 = 0$, which implies that $n = 2k+1$, Now $n = 1 + k^2$, so $k = 2$. There is such a graph, namely the 5-cycle.

If $4k-3$ is a square of an integer, let

$$l = \sqrt{4k-3}, \qquad l^2 = 4k-3$$

We deduce that l is odd. Applying the fact that the sum of the eigenvalues is 0, we deduce that

$$k - \frac{n-1}{2} + (2s - (n-1))\frac{l}{2} = 0.$$

Then using $k = (l^2+3)/4$ and $(n-1)/2 = k^2/2 = (l^2+3)^2/16$, we obtain

$$2s - (n-1) = \frac{l^4 - 2l^2 - 15}{8l}$$

and hence l divides $l^4 - 2l^2 - 15$ evenly. But then l divides 15 evenly and hence $l = 1, 3, 5, 15$ are the only possibilities yielding $k = 1, 3, 7, 57$ as the only possibilities. The case $k = 1$ makes no sense (a single edge is not of diameter 2). The case $k = 3$ yields the Petersen graph on $10 = 3^2 + 1$ vertices. The case $k = 7$ yields the Hoffman-Singleton graph on $50 = 7^2 + 1$ vertices. The case $k = 57$ allows for the possibility of a Moore graph on $3250 = 57^2 + 1$ vertices. The existence or non existence of such a graph is unknown.

Summarizing our work, we have that here are Moore graphs regular of degree 2 (5-cycle) on 5 vertices, degree 3 (Petersen graph) on 10 vertices, degree 7 (Hoffman-Singleton graph) on 50 vertices and there may be Moore graphs regular of degree 57 on 3250 vertices.