Big new concepts in MATH 223 include a *vector space*, *linear independence* (or *linear dependence*), and *dimension*.

Definition *A set* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *of* k *vectors is said to be* linearly dependent *if there are coefficients* $a_1, a_2, \ldots, a_k$ not all zero *such that* $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$.

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Note that $\mathbf{0}$ is a linearly dependent set, since $1 \cdot \mathbf{0} = \mathbf{0}$.

Definition *For a vector space* V*, a* basis *is a linearly independent set of vectors* S *so that* $span(S) = V$.

There would be two ways to find a basis. Either begin with a spanning set, and reduce if there are any dependencies. Alternatively build the basis from the ground up as a linearly independent set contained in V

Theorem *Any basis for a vector space* V *has the same cardinality.*

Proof: We let $B_1 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ and $B_2 = \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_\ell\}$ be two bases for V. Assume that $\ell > k$. Now because B_1 is a basis, then any vector in V is a linear combination of vectors in B_1 and so we may write, without strange names for the coefficients, that

$$\mathbf{v}_j = \sum_{i=1}^{k} a_{ij}\mathbf{u}_i$$

Thus if we let $A = (a_{ij})$ be the matrix with these entries then the jth column of A corresponds to $\mathbf{v}_j$. now because $k < \ell$, then when we solve $A\mathbf{x} = \mathbf{0}$, we will have at most k pivot variables and hence at least $\ell - k > 0$ free variables and hence an $\mathbf{x} \neq \mathbf{0}$ with $A\mathbf{x} = \mathbf{0}$.

We think of $\mathbf{x}$ as yielding a linear combination of the $\mathbf{v}_j$'s yielding the zero vector, which would be a contradiction. Let $\mathbf{x} = (x_1, x_2, \ldots, x_\ell)^T$. Then

$$\sum_{j=1}^{\ell} x_j\mathbf{v}_j = \sum_{j=1}^{\ell} x_j \left(\sum_{i=1}^{k} a_{ij}\mathbf{u}_i\right)$$

$$= \sum_{i=1}^{k} \left(\sum_{j=1}^{\ell} a_{ij}x_j\right) \mathbf{u}_i = \sum_{i=1}^{k} 0 \cdot \mathbf{u}_i = \mathbf{0}$$

This has verified that B_2 is linearly dependent, a contradiction to B_2 being a basis and hence we conclude that $k = \ell$.

Definition The *dimension* of a vector space V is the cardinality of any basis for V.

The dimension of $\mathbf{R}^t$ is t since we can identify a basis of $\mathbf{R}^t$ as $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_t\}$ where $\mathbf{e}_i$ denote the vector with a 1 in the ith coordinate and 0's elsewhere. Any vector space V contained in $\mathbf{R}^t$ has dimension at most t. (How should you show that the dimension is at most t? Assume you have $t+1$ linear independent vectors in V and derive a contradiction). Thus *dimension* is being used as a piece of mathematical terminology for vector spaces in the context of bases and does not refer some English meaning of dimension (length, width, height?). Maybe we would have been better to have a separate term but this is not standard.

We have finally shown that we are in a three dimensional world.