MATH 223 Complex Numbers II Richard Anstee

Let $z = a + bi$ and $w = c + di$. We defined

$$zw = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

There are some interesting observations about this product. It is often the case that complex numbers are viewed as points in the *Argand Plane*, so that z is placed at the point $(Re(z), Im(z))$. We note $z\overline{z} = a^2 + b^2$. In the argand plane it is natural to define the *modulus* of z

$$|z| = \sqrt{a^2 + b^2}$$

which is the same as $z\overline{z} = |z|^2$ (which you shall see in the context of inner product spaces). We check

$$|z||w| = (a^2 + b^2)(c^2 + d^2) = a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$$

and with $zw = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$, we have

$$|zw| = (ac-bd)^2+(ad+bc)^2 = a^2c^2+b^2d^2-2abcd+a^2d^2+b^2c^2+2abcd = a^2c^2+a^2d^2+b^2c^2+b^2d^2 = |z||w|$$

This is quite surprising. Now we also can think of an angle θ associated with z in the argand plane namely the angle between the Re axis and the vector (2-tuple) joining the origin (0+0i) with the point z. So

$$z = |z|(\cos(\theta) + i\sin(\theta)) = |z|e^{i\theta}$$

where

$$\cos(\theta) = \frac{a}{\sqrt{a^2 + b^2}}, \quad \sin(\theta) = \frac{b}{\sqrt{a^2 + b^2}}$$

With this notation we say that the *argument* of z is

$$\arg(z) = \theta.$$

Now what about $\arg(zw)$? Assume $\arg(w) = \phi$. We could write $z = |z|e^{i\theta}$ and $w = |w|e^{i\phi}$ and so

$$zw = |z||w|e^{i(\theta+\phi)}$$

which yields $\arg(zw) = \theta + \phi$. Thus multiplying two complex numbers multiplies their moduli and adds their arguments.

Alternatively

$$\cos(\theta) = \frac{a}{\sqrt{a^2 + b^2}}, \quad \sin(\theta) = \frac{b}{\sqrt{a^2 + b^2}}, \quad \cos(\phi) = \frac{c}{\sqrt{c^2 + d^2}}, \quad \sin(\phi) = \frac{d}{\sqrt{c^2 + d^2}}$$

We have by our angle sum formulas (from the first assignment!)

$$\cos(\theta + \phi) = \frac{ac - bd}{\sqrt{(a^2 + b^2)(c^2 + d^2)}}, \sin(\theta + \phi) = \frac{ad + bc}{\sqrt{(a^2 + b^2)(c^2 + d^2)}}.$$

Thus $\theta + \phi = \arg(zw)$.

This has some interesting consequences. Note that if $\arg(z) = t$ and $|z| = 1$, then the $Re(z^n)$, $Im(z^n)$ traces out repeated rotation by t.