If we have a set of vectors \( \{u_1, u_2, \ldots, u_k\} \) where we set \( U = \text{span} \{u_1, u_2, \ldots, u_k\} \), it is natural to express any vector \( u \in U \) as a linear combination of the vectors \( u_1, u_2, \ldots, u_k \), namely
\[
    u = c_1u_2 + c_2u_2 + \cdots + c_ku_k
\]
where we think of \( c_1, c_2, \ldots, c_k \) as the coordinates of \( u \) with respect to the spanning set \( \{u_1, u_2, \ldots, u_k\} \). Now if \( \{u_1, u_2, \ldots, u_k\} \) is linearly independent, then the coordinates behave as we would hope, namely they are unique.

**Theorem 1** If the set \( \{u_1, u_2, \ldots, u_k\} \) is linearly independent, then for each vector \( u \in U = \text{span} \{u_1, u_2, \ldots, u_k\} \), there are unique numbers \( c_1, c_2, \ldots, c_k \) (the coordinates) such that \( u = c_1u_2 + c_2u_2 + \cdots + c_ku_k \).

**Proof:** The existence of numbers \( c_1, c_2, \ldots, c_k \) follows from the fact that \( u \in U = \text{span} \{u_1, u_2, \ldots, u_k\} \). Assume
\[
    u = c_1u_2 + c_2u_2 + \cdots + c_ku_k
\]
and
\[
    u = d_1u_2 + d_2u_2 + \cdots + d_ku_k
\]
Then by subtracting the two equations we obtain
\[
    0 = (c_1 - d_1)u_2 + (c_2 - d_2)u_2 + \cdots + (c_k - d_k)u_k.
\]
Since the set \( \{u_1, u_2, \ldots, u_k\} \) is linearly independent, then we deduce that \( c_1 - d_1 = 0, c_2 - d_2 = 0, \ldots, c_k - d_k = 0 \) and hence \( c_1 = d_1, c_2 = d_2, \ldots, c_k = d_k \).

Thus if we have a \( k \)-dimensional vector space than we can coordinatize the vectors as elements of \( \mathbb{R}^k \). Consider the following 4 vectors.

\[
    v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}
\]

We can verify that \( U = \text{span} \{v_1, v_2, v_3, v_4\} = \text{span} \{v_1, v_2\} \) noting that \( v_3 = -7v_1 + 4v_2 \) and \( v_4 = -5v_1 + 4v_2 \). Indeed \( \dim(U) = 2 \). While \( U \subseteq \mathbb{R}^3 \) it is natural to consider \( U \) as a 2-dimensional vector space and in fact we can write our vectors in blue coordinates with respect to the basis \( v_1, v_2 \) of \( U \).

\[
    \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ is } \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \text{ is } \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} \text{ is } \begin{bmatrix} -7 \\ 4 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \text{ is } \begin{bmatrix} -5 \\ 4 \end{bmatrix}.
\]

A somewhat different example is from the assignment. Let \( W = \text{span} \{\cos^2(x), \sin^2(x)\} \). We deduce that \( \{\cos^2(x), \sin^2(x)\} \) is a basis for \( W \) so we can coordinatize with respect to this basis.

\[
    \cos^2(x) \text{ is } \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \sin^2(x) \text{ is } \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad 2 \text{ is } \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \cos(2x) \text{ is } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]

As a vector space over \( \mathbb{R} \) we can think of \( W \) as \( \mathbb{R}^2 \). Of course as functions, there are more properties. We can’t differentiate a vector but we can differentiate \( \cos^2(x) \).

A student in MATH 223 in 2015 said that \( U \) and \( W \) were thinly veiled examples of \( \mathbb{R}^2 \). And of course similarly we think of a vector space \( X \), with \( \dim(X) = k \) and \( \mathbb{R} \) as the scalar field, as a thinly veiled example of \( \mathbb{R}^k \). To make this precise consider the following definition.
Definition 2  Given two vector spaces $U, V$ over the same field $F$, we say that $U$ and $V$ are isomorphic if there is a bijective map $h : U \to V$ with $h(0) = 0$ (the first 0 is in $U$ and the second 0 is in $V$) and with the property that for any $x, y \in U$, we have $h(x + y) = h(x) + h(y)$ and for any $c \in F$, $h(cx) = c \cdot h(x)$.

Remember that the isomorphism need not preserve other properties of the elements of $U$ and $V$ that are not associated with being a vector space.

Theorem 3 If $U$ and $V$ are vector spaces over the same field and $\dim(U) = \dim(V)$ then $U$ and $V$ are isomorphic.

Proof: Let $k = \dim(U) = \dim(V)$. Assume $k > 0$. Let $U$ have basis $u_1, u_2, \ldots, u_k$ and $V$ has basis $v_1, v_2, \ldots, v_k$. Then define $h(u_i) = v_i$ and extend to all vectors of $U$ by linearity; namely for $u = \sum_{i=1}^{k} a_i u_i$ and so define $h(u) = \sum_{i=1}^{k} a_i v_i$. We easily show that $h$ is a bijection and $h^{-1}(v_i) = u_i$.

When $0 = \dim(U) = \dim(V)$, then each consists of just the zero vector and so the isomorphism is easy.  

The following is an important application of dimension.

Theorem 4 An $m \times m$ matrix $A$ is diagonalizable if and only if there is a basis of $\mathbb{R}^m$ consisting of eigenvectors of $A$.

Proof: If $A$ is diagonalizable then there is a diagonal matrix $D$ and an invertible matrix $M$ with $AM = MD$. But then each column of $M$ is an eigenvector of $A$ (no column of $M$ can be 0 since $M$ is invertible. And since $M$ is invertible, the only solution to $Mx = 0$ is $x = 0$. Thus the $m$ columns of $M$ are linearly independent. But we note the columns of $M$ are contained in $\mathbb{R}^m$. Thus the dimension of the column space of $M$ is $m$ and so the column space of $M$ is equal to $\mathbb{R}^m$.

If there is a basis of $\mathbb{R}^m$ say $\{v_1, v_2, \ldots, v_m\}$ then if we form the matrix $M$ whose columns are the $v_i$’s then $M$ is invertible. If $Av_i = \lambda_i v_i$, then we have $AM = MD$ where the $i$th diagonal entry is $\lambda_i$.

We will add some more detail to this theorem as course progresses.