We seek a determinant function $\det : \mathbf{R}^{n\times n} \to \mathbf{R}$ that satisfies various natural properties.

- $\det I = 1$
- If B is obtained by multiplying row i of A by t then $\det(B) = t\cdot\det(A)$
- If B is obtained from A by interchanging row i and row j then $\det(B) = -\det(A)$
- If B is obtained from A by adding a multiple of row i to row j then $\det(B) = \det(A)$
- $\det(AB) = \det(A)\det(B)$
- $\det(A) \neq 0$ if and only if A has an inverse
- $\det(A^T) = \det(A)$
- $\det(A)$ measures some volume: $|\det(A)|$ is the volume of the parallelepiped formed by the column vectors of A

The idea is to give a specific function and verify that it has the desired properties. For convenience, use the notation M_{ij} to denote the matrix obtained from A by deleting row i and column j. We define

$$\det(A) = (-1)^{1+1}a_{11}\det M_{11} + (-1)^{1+2}a_{12}\det M_{12} + \cdots + (-1)^{1+n}a_{1n}\det M_{1n} \tag{1}$$

This is called *expansion about the first row* because one of our goals is to show that the following formulas are equivalent

$$\det(A) = (-1)^{i+1}a_{i1}\det M_{i1} + (-1)^{i+2}a_{i2}\det M_{i2} + \cdots + (-1)^{i+n}a_{in}\det M_{in}$$

and

$$\det(A) = (-1)^{1+j}a_{1j}\det M_{2j} + (-1)^{2+j}a_{2j}\det M_{2j} + \cdots + (-1)^{n+j}a_{nj}\det M_{nj}$$

We embark on showing some facts/lemmas that can be deduce from (1). The numbering of the lemmas is not important. We organize our results to obtain the results involving the elementary matrices which will be used to prove $\det(AB) = \det(A)\det(B)$.

Lemma 1 *Let A be an $n\times n$ matrix. If A has a row of 0's or a column of 0's then* $\det(A) = 0$.

Proof: We prove this by induction on n with the base case for $n = 2$ being easy. ■

Lemma 2 *Let A be an $n\times n$ triangular matrix with diagonal entries $d_1, d_2, \ldots, d_n$. Then*

$$\det(A) = \prod_{i=1}^{n} d_i$$

Proof: We apply expansion about the first row and discover that M_{1j} has a column of 0's unless $j = 1$ and so we can apply the previous lemma. We apply induction on n for $\det(M_{11})$. ■

Corollary 3 $\det(I) = 1$.

Lemma 4 *Let $E(i,j)$ denote the matrix which interchanges row i and j.Then* $\det(E(i,j)A) = -\det(A)$

Proof: We leave the proof for $i = 1$,$j = 1$ as a separate and rather technical argument. The rest we can prove by induction. This is easy for $E(i, j)$ where both $i, j \geq 2$ but $E(1, j)$ for $j \geq 3$ is handled differently. We have that $E(1, 2)E(2, j)E(1, 2)A = E(1, j)A$ and that is how we apply induction under the assumption that we have proven $\det(E(1, 2)A) - \det(A)$ by an independent argument. ■

Corollary 5 $\det(E(i, j)) = -1$

Proof: Use $A = I$ in $\det(E(i, j)A) = -\det(A)$. ■

Lemma 6 *Let $D(i, t)$ denote the matrix which multiplies row i by t. Thus* $\det(D(i, t)) = t$ *by Lemma 2 and also* $\det(D(i, t)A) = t \cdot \det(A)$.

Proof: For $i = 1$, this is straightforward by (1). For $i > 1$, then use induction on n to deduce that the $1, j$ minor of $D(i, t)A$ is $D(i - 1, t)M_{ij} = t \cdot \det(M_{1j})$ where M_{ij} is $1, j$ minor of A. ■

Corollary 7 $\det(D(i, t)) = t$.

Proof: $D(i, t)I = D(i, t)$ and $\det(I) = 1$. ■

Lemma 8 *Let A have two identical rows. Then* $\det(A) = 0$.

Proof: Let the two identical rows be i, j. Then $E(i, j)A = A$. Then by the above lemma, $\det(A) = \det(E(i, j)A) = -\det(A)$ and so $\det(A) = 0$. ■

Lemma 9 *Let A, B, C and i be given. Assume A, B, C are identical with the possible exception of row i for which row i of A plus row i of B is equal to row i of C. Then* $\det(A) + \det(B) = \det(C)$.

Proof: This is immediate by (1) when $i = 1$. For $i > 1$, apply induction on n. Let M_{1j} be the minor from A, M'_{1j} be the minor from B and M''_{1j} be the minor from C. Then the $(i-1)$st row of M_{1j} plus the $(i-1)$st row of M'_{1j} is equal to the $(i-1)$st row of M''_{ij} where the $(n - 1) \times (n - 1)$ minors are otherwise the same. Thus $\det(M_{1j}) + \det(M'_{1j}) = \det(M''_{1j})$. Then we can obtain $\det(A) + \det(B) = \det(C)$. ■

Lemma 10 *Let i, j, t be given and let $F(i, j, t)$ denote the elementary matrix corresponding to adding t times row i to row j. Then* $\det(F(i, j, t)A) = \det(A)$.

Proof: Apply the previous lemma noting that row j of $F(i, j, t)A$ is row j of A plus t times row i of A. Thus we have two matrices A, B with B coming from A by replacing row j by t times row i. Now $\det(B) = 0$, since row j of B is t times row i of B. If $t = 0$ we use Lemma 1. Or perhaps more obviously we note that $F(i, j, 0)A = A$.

If $t \neq 0$, we apply Lemma 6 to remove the factor t and then use Lemma 8. Now $\det(F(i, j, t)A) = \det(A) + \det(B) = \det(A)$. ■

Corollary 11 $\det(F(i, j, t)) = 1$.

Proof: Use $A = I$ in Lemma 10. ■

Theorem 12 *A is invertible if and only if* $\det(A) \neq 0$ *if and only if there exists an* $\mathbf{x} \neq \mathbf{0}$ *with* $A\mathbf{x} = \mathbf{0}$.

Proof: Apply our row reductions to reduce A to staircase pattern B by Gaussian Elimination. From Corollary 5, Corollary 7, Corollary 11, $\det(A) \neq 0$ if and only if $\det(B) \neq 0$. Now $\det(B) \neq 0$ will yield (by Lemma 2) that B must be an (upper) triangular matrix with non zero elements on the diagonal. By our previous work, this implies A has an inverse. Also $\det(B) = 0$ will yield that there will be a 0 on the diagonal and so in the staircase pattern, there will be fewer corner/pivot variables than n and so there will be at least one free variable. Then we can find a vector $\mathbf{x} \neq \mathbf{0}$ with $A\mathbf{x} = \mathbf{0}$ which implies that A has no inverse. ■

Theorem 13 $\det(AB) = \det(A)\det(B)$

Proof: We first split into two cases, depending on whether A is invertible or not.
Case 1: A is invertible.

If A is invertible then we can express A as a product of our elementary matrices; $A = E_1E_2E_3\cdots E_t$. We noted this after we used Gaussian Elimination to find A^{-1}. Now proceed using Lemma 4, Lemma 6, Lemma 10 as needed for elementary matrices.

$$AB = E_1E_2E_3\cdots E_tB = (E_1(E_2(E_3\cdots(E_tB)\cdots)))$$

Thus (by induction)

$$\det(AB) = \det(E_1)\det(E_2)\cdots\det(E_t)\det(B)$$

and also (by the same induction)

$$\det(A) = \det(E_1(E_2(E_3\cdots(E_{t-1}E_t)\cdots))) = det(E_1)\det(E_2)\cdots\det(E_t).$$

This yields $\det(AB) = \det(A)\det(B)$.
Case 2: A is not invertible.

Thus $\det(A) = 0$. If we have $\det(AB) \neq 0$, then there exists a matrix $M = (AB)^{-1}$ so that $ABM = I$ from which we have $BM = A^{-1}$, a contradiction. So $\det(AB) = 0 = \det(A)\det(B)$. ■

Theorem 14 *Let i be given with $1 \leq i \leq n$. Then we can compute $\det(A)$ by expansion about the ith row:*

$$\det(A) = (-1)^{i+1}a_{i1}\det M_{i1} + (-1)^{i+2}a_{i2}\det M_{i2} + \cdots + (-1)^{i+n}a_{in}\det M_{in}$$

Proof: Simply apply the row interchanges $(i, i-1)$, $(i-2, i-1)$, ..., $(1, 2)$ which brings row i up to the first row and changes the determinant by $(-1)^{i-1}$. Now apply (1) to the new matrix. ■

Theorem 15 $\det(A^T) = \det(A)$

Proof: We could simply rewrite what we have done using elementary column operations and using multiplication on the left. A simpler approach, assuming A is invertible, is to write A as a product of elementary matrices so that $A = E_1E_2E_3\cdots E_t$. Then $A^T = (E_t)^T(E_{t-1})^T\cdots(E_2)^T(E_1)^T$. We then need only verify that $\det((E_j)^T) = \det(E_j)$. For the interchange matrix $E(i,j)$, we have $E(i,j)^T = E(i,j)$ (we obtain $E(i,j)$ by starting with the identity I and then interchanging rows i and j which is the same as interchanging columns i and j of the identity). For the multiplication matrix $D(i,t)$ we deduce $(D(i,t))^T = D(i,t)$. For the matrix adding a multiple of one row to another we have $(F(i,j,t))^T = F(j,i,t)$ and so $\det((F(i,j,t))^T) = \det(F(j,i,t)) = 1 = \det(F(i,j,t)$. ■

I was offered the following explanation using concepts of Pure Mathematics (concepts that are not the subject of this course). They said *The dualization of Vector Spaces is an exact functor*!?

Theorem 16 *Let j be given with $1 \leq j \leq n$. Then we can compute* $\det(A)$ *by expansion about the jth column:*

$$\det(A) = (-1)^{1+j} a_{1j} \det M_{2j} + (-1)^{2+j} a_{2j} \det M_{2j} + \cdots + (-1)^{n+j} a_{nj} \det M_{nj}$$

Proof: Simply apply Lemma 15 and then use expansion about the jth row of A^T. ■

This particular approach to determinants starts with a very concrete expression for the determinant (expansion about the first row) and then determines various properties that follow from that expression. There are other approaches to determinants. One interesting approach that Apostol used was to give 4 axioms that a function such as the determinant should satisfy and then verify that there is a unique function that satisfies then and that it can be computed using (1).

Consider a function f of n vectors in $\mathbf{R}^n$. We say that f is called a determinant function of order n if it satisfies the following axioms.

Axiom 1 (Homogeneity in each row) $f(\mathbf{x}_1, \mathbf{x}_2, \ldots, t \cdot \mathbf{x}_k, \ldots) = t \cdot f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \ldots)$.

Axiom 2 (additivity in each row) $f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k + \mathbf{y}, \ldots) = f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \ldots) + f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{y}, \ldots)$

Axiom 3 (vanishes if two rows are equal) $f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \ldots) = 0$ if $\mathbf{x}_k = \mathbf{x}_\ell$ for $k \neq \ell$.

Axiom 4 (determinant of I is 1) $f(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k, \ldots) = 1$ where $\mathbf{e}_i$ is the vector with a 1 in row i and zeros elsewhere.

The Apostol calculus book carries through to show that these 4 axioms determine a unique function that matches our determinant given in (1).