MATH 223: Eigenvalues and Eigenvectors.

Consider a 2×2 matrix A. When a vector $\mathbf{v}$ satisfies

$$\mathbf{v} \neq \mathbf{0},$$

$$A\mathbf{v} = \lambda \mathbf{v}$$

then we say that $\mathbf{v}$ is an *eigenvector* of A of *eigenvalue* λ. We note

$$A(k\mathbf{v}) = k(A\mathbf{v}) = k(\lambda\mathbf{v}) = \lambda(k\mathbf{v}),$$

which says that non zero multiples of eigenvectors yield more eigenvectors of the same eigenvalue. Let us first consider the geometric transformations we previously mentioned. An eigenvector will correspond to a direction that is fixed (or reversed) by the transformation.

$$D(2,3) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

will have $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as an eigenvector of eigenvalue 2 and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as an eigenvector of eigenvalue 3. The identity matrix I has the property that any non zero vector $\mathbf{v}$ is an eigenvector of eigenvalue 1.

The rotation matrix $R(\theta)$ has no eigenvectors, by the geometric reasoning that no directions are preserved, unless $\theta = 0, \pi$. There will be no (real) roots of the quadratic.

The *shear* matrix $G_{12}(\gamma) = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}$ has $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as an eigenvector of eigenvalue 1 but no other eigenvectors (other than multiples) for $\gamma \neq 0$.

The following analysis is critical in seeking eigenvectors and eigenvalues:

$$\text{there exists a } \mathbf{v} \text{ with } A\mathbf{v} = \lambda\mathbf{v}; \quad \mathbf{v} \neq \mathbf{0}$$

$$\text{if and only if there exists a } \mathbf{v} \text{ with } A\mathbf{v} = \lambda I\mathbf{v}; \quad \mathbf{v} \neq \mathbf{0}$$
$$\text{if and only if there exists a } \mathbf{v} \text{ with } (A - \lambda I)\mathbf{v} = \mathbf{0}; \quad \mathbf{v} \neq \mathbf{0}$$
$$\text{if and only if } \det(A - \lambda I) = 0$$

Now

$$\det(A - \lambda I) = \det(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}) = \lambda^2 - (a+d)\lambda + (ad - bc)$$
$$= \lambda^2 - \mathrm{tr}(A)\lambda + \det(A),$$

which (for 2×2 matrices) is a quadratic function in λ and whose roots you can seek by standard methods.

Sample computation

$$A = \det(\begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix})$$

$$\det(A - \lambda I) = \det(\begin{bmatrix} .7-\lambda & .3 \\ 2 & -\lambda \end{bmatrix})$$

$$= (.7-\lambda)(-\lambda) - .3\times 2$$
$$= \frac{1}{10}(10\lambda^2 - 7\lambda - 6)$$
$$= \frac{1}{10}(5\lambda-6)(2\lambda+1)$$

Thus we have two eigenvalues $\lambda = \frac{6}{5}, \frac{-1}{2}$.

For $\lambda = \frac{6}{5}$, we solve $(A - \frac{6}{5}I)\mathbf{v} = \mathbf{0}$ for $\mathbf{v} \neq \mathbf{0}$:

$$(A - \frac{6}{5}I)\mathbf{v} = \begin{bmatrix} -.5 & .3 \\ 2 & -1.2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The vector $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ works as an eigenvalue of A of eigenvalue $\frac{6}{5}$. We check

$$\begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3.6 \\ 6 \end{bmatrix} = \frac{6}{5}\begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

For $\lambda = \frac{-1}{2}$, we solve $(A - \frac{-1}{2}I)\mathbf{v} = \mathbf{0}$ for $\mathbf{v} \neq \mathbf{0}$:

$$(A - \frac{-1}{2}I)\mathbf{v} = \begin{bmatrix} 1.2 & .3 \\ 2 & .5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The vector $\mathbf{v} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ works as an eigenvalue of A of eigenvalue $\frac{-1}{2}$. We check

$$\begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} -.5 \\ 2 \end{bmatrix} = \frac{-1}{2}\begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

Note that we will always succeed in finding an eigenvector (a non zero vector) assuming our eigenvalue λ has $\det(A - \lambda I) = 0$.

The origin of this matrix was a model of bird populations. Let

$$x_n = \text{ no. of adults in year } n,$$

$$y_n = \text{ no. of juveniles in year } n.$$

We have a matrix equation to represent changes from year to year. We have 30% of the juveniles survive to become adults, 70% of the adults survive a year, and each adult has 2 offspring (juveniles). We have this information summarized in a matrix equation:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix}.$$

We deduce by induction, that

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$