MATH 223: Eigenvalues and Eigenvectors (and Birds). Richard Anstee

The following idea is important in a variety of contexts in this course. For a matrix A, assume we have two eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ of eigenvalues λ_1, λ_2. Form the matrix

$$M = [\mathbf{v}_1\, \mathbf{v}_2].$$

We have the matrix equation

$$AM = MD$$

where

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Now make the assumption that M is invertible. This is a non trivial assumption. For us, it is true as long as $\mathbf{v}_1 \neq k\mathbf{v}_2$ for any k.

We can verify this to be true if $\lambda_1 \neq \lambda_2$. Assume $\mathbf{v}_1 = k\mathbf{v}_2$ and get a contradiction:

$$A\mathbf{v}_1 = A(k\mathbf{v}_2) = kA(\mathbf{v}_2) = k\lambda_2\mathbf{v}_2 = \lambda_2\mathbf{v}_1,$$

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1.$$

We conclude that $\lambda_2\mathbf{v}_1 = \lambda_1\mathbf{v}_1$, i.e. $(\lambda_1 - \lambda_2)\mathbf{v}_1 = \mathbf{0}$ and so, with $\mathbf{v}_1 \neq \mathbf{0}$, $\lambda_1 - \lambda_2 = 0$ and so $\lambda_1 = \lambda_2$ which is a contradiction. Thus $\mathbf{v}_1 \neq k\mathbf{v}_2$ for any k.

Now

$$AM = MD \text{ means } M^{-1}AM = D \text{ and } A = MDM^{-1}.$$

In our case with the Leslie Matrix and birds

$$A = \begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} 3 & 1 \\ 5 & -4 \end{bmatrix}, \qquad M^{-1} = \begin{bmatrix} \frac{4}{17} & \frac{1}{17} \\ \frac{5}{17} & \frac{-3}{17} \end{bmatrix}, \qquad D = \begin{bmatrix} \frac{6}{5} & 0 \\ 0 & \frac{-1}{2} \end{bmatrix}$$

Now we have $A = MDM^{-1}$ and so

$$A^2 = MDM^{-1}MDM^{-1} = MD(M^{-1}M)DM^{-1} = MD^2M^{-1},$$

$$A^3 = MDM^{-1}MDM^{-1}MDM^{-1} = MD(M^{-1}M)D(M^{-1}M)DM^{-1} = MD^3M^{-1},$$

$$A^n = MD^nM^{-1}.$$

It is straightforward to compute

$$D^n = \begin{bmatrix} (\frac{6}{5})^n & 0 \\ 0 & (\frac{-1}{2})^n \end{bmatrix},$$

hence

$$A^n = \begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix}^n = \begin{bmatrix} 3 & 1 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} (\frac{6}{5})^n & 0 \\ 0 & (\frac{-1}{2})^n \end{bmatrix} \begin{bmatrix} \frac{4}{17} & \frac{1}{17} \\ \frac{5}{17} & \frac{-3}{17} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12}{17}(1.2)^n + \frac{5}{17}(-.5)^n & \frac{3}{17}(1.2)^n - \frac{3}{17}(-.5)^n \\ \frac{20}{17}(1.2)^n - \frac{20}{17}(-.5)^n & \frac{5}{17}(1.2)^n + \frac{12}{17}(-.5)^n \end{bmatrix}.$$

Thus

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{12}{17}(1.2)^n + \frac{5}{17}(-.5)^n \\ \frac{20}{17}(1.2)^n - \frac{20}{17}(-.5)^n \end{bmatrix} \approx \begin{bmatrix} \frac{12}{17}(1.2)^n \\ \frac{20}{17}(1.2)^n \end{bmatrix},$$

where we are using the fact that $\lim_{n\to\infty}(-.5)^n = 0$. One aspect of the result is that the population is growing 20% a year and also the ratio of adults to juveniles is approximately $3:5$ in a stable population. A ratio sufficiently far from $3:5$ would alert the biologist to the likelihood of the population having undergone some environmental disturbance in the recent past.

There are many important applications of eigenvectors/eigenvalues including computing powers of a matrix. We intend to compute the nth fibonacci number this way.