Let $A$ be an $m \times n$ matrix. Each column is a vector in $\mathbb{R}^m$ and each row, when interpreted as a column, is a vector in $\mathbb{R}^n$. Let $A_i$ denote the $i$th column of $A$. We define the column space of $A$, denoted $\text{colsp}(A)$, as the span $\{A_1, A_2, \ldots, A_n\}$. Similarly we define the row space of $A$, denoted $\text{rowsp}(A)$ as the span of the rows of $A$, when interpreted as column vectors in $\mathbb{R}^n$.

We have already noted that for $x = (x_1, x_2, \ldots, x_n)^T$, we have $Ax = \sum_{i=1}^n x_i A_i \in \text{colsp}(A)$. A consequence is that $\text{colsp}(A) = \text{Im}(f)$ where we use $\text{Im}(f)$ to denote the image space (or range) of the linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$ given by $f(x) = Ax$.

We have previously noted the following

**Proposition 1** Let $A$ be an $m \times n$ matrix.

(a) If $M$ is an $m \times m$ matrix then $\{x : Ax = 0\} \subseteq \{x : MAx = 0\}$

(b) If $M$ is an invertible $m \times m$ matrix, then $\{x : Ax = 0\} = \{x : MAx = 0\}$

We proved (b) at the beginning of the course (in the context of $\{x : Ax = b\}$ but you can specialize to $b = 0$). Results related to (a) are often used in midterms.

We can also prove results for $\text{rowsp}(A)$ by simply using $\text{rowsp}(A) = \text{colsp}(A^T)$ but it makes sense to use the staircase pattern obtained by applying Gaussian elimination to $A$.

**Proposition 2** Let $A$ be an $m \times n$ matrix.

(a) If $M$ is an $m \times m$ matrix then $\text{rowsp}(MA) \subseteq \text{rowsp}(A)$

(b) If $M$ is an invertible $m \times m$ matrix then $\text{rowsp}(MA) = \text{rowsp}(A)$

Consider the following example which we imagine was obtained by Gaussian elimination.

$$A = \begin{bmatrix} 2 & -2 & 0 & 2 & 1 & 0 & 0 \\ 4 & -4 & 0 & 4 & 3 & 2 & 2 \\ 2 & -1 & 3 & 4 & 1 & 1 & 2 \\ 2 & 0 & 6 & 6 & 2 & 4 & 8 \end{bmatrix}$$

With $E$ invertible we obtain

$$EA = \begin{bmatrix} 2 & -2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Any linear dependence among the columns such as $y_1 A_1 + y_2 A_2 + \cdots + y_n A_n = 0$ with $y = (y_1, y_2, \ldots, y_n)^T$ yields a solution to $Ay = 0$ and vice versa namely any $y = (y_1, y_2, \ldots, y_n)^T$ with $Ay = 0$ yields $y_1 A_1 + y_2 A_2 + \cdots + y_n A_n = 0$. Let $I$ denote a subset of $\{1, 2, \ldots, n\}$, namely a subset of the column indices. Let $A_i$ denote the $i$th column of $A$ so that $(EA)_i$ denotes the $i$th column of $EA$. We deduce the following using Proposition 1.

**Proposition 3** Let $A, E$ be given with $E$ being invertible. The set of columns $\{A_i : i \in I\}$ is linearly dependent if and only if the set of columns $\{(EA)_i : i \in I\}$ is linearly dependent.
Corollary 4 Let $A, E$ be given with $E$ being invertible. It then follows that the set of columns $\{A_i : i \in I\}$ is linearly independent if and only if the set of columns $\{(EA)_i : i \in I\}$ is linearly independent and hence the set of columns $\{A_i : i \in I\}$ forms a basis for $\text{colsp}(A)$ if and only if the set of columns $\{(EA)_i : i \in I\}$ forms a basis for $\text{colsp}(EA)$.

When we look at staircase patterns $EA$, where $E$ is invertible, it is easy to identify linearly independent columns of $EA$ whose span is $\text{colsp}(EA)$. Given that the sets of columns that are linearly dependent in $A$ are precisely those that are linearly dependent in $EA$, then it is also true that those that are linearly independent in $A$ are precisely those that are linearly independent in $EA$. Hence a set of columns of $A$ yielding a column basis for $\text{colsp}(A)$ will correspond to a set of columns of $EA$ yielding a column basis for $\text{colsp}(EA)$. Note that the idea is that the 1st, 2nd and 5th columns of $EA$ yield a column basis for $\text{colsp}(EA)$ if and only if the 1st, 2nd and 5th columns of $A$ yield a column basis for $\text{colsp}(A)$. It is straightforward to deduce that a basis for $\text{colsp}(EA)$ are columns 1, 2 and 5:

$$
\begin{bmatrix}
2 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
-2 \\
1 \\
0
\end{bmatrix},
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
$$

and so, by Corollary 4, a basis for $\text{colsp}(A)$ is

$$
\begin{bmatrix}
2 \\
4 \\
2
\end{bmatrix},
\begin{bmatrix}
-2 \\
-4 \\
-1
\end{bmatrix},
\begin{bmatrix}
1 \\
3 \\
1
\end{bmatrix}
$$

There are other choices for column bases but it is easiest to chose the columns of $A$ whose corresponding columns in $EA$ contain the pivots.

We can now use the (relatively) easy observation that the nonzero rows of $EA$ form a basis for $\text{rowsp}(EA)$, namely a basis for $\text{rowsp}(EA)$ is $\{(2, -2, 0, 2, 1, 0, 0)^T, (0.1.3.2.0.1.3)^T, (0, 0, 0, 0, 1, 3, 2)^T\}$. Combine this with Proposition 2 with $E$ being invertible and we have that the nonzero rows of $EA$ are also a basis for $\text{rowsp}(A)$.

We have defined $\text{rowsp}(A) = \text{span}\{(2, -2, 0, 2, 1, 0, 0)^T, (4, -4, 0, 4, 3, 2, 2)^T, (2, -1, 3, 4, 1, 1, 3)^T, (2, 0, 6, 6, 2, 4, 8)^T\}$. With $E$ being invertible we have $\text{rowsp}(A) = \text{rowsp}(EA)$ and so a basis for $\text{rowsp}(A)$ is $\{(2, -2, 0, 2, 1, 0, 0)^T, (0, 1, 3, 2, 0, 1, 3)^T, (0, 0, 0, 0, 1, 3, 2)^T\}$. Please note that $E$ being invertible does not mean that the first 3 rows of $A$ form a basis for $\text{rowsp}(A)$, although it is possible.

Theorem 5 $\dim(\text{rowsp}(A)) = \dim(\text{colsp}(A))$.

Proof: We have $\dim(\text{rowsp}(A))$ being equal to the number of non-zero rows of $EA$ and hence the number of pivots and we have $\dim(\text{colsp}(A))$ being equal to the size of a basis for $\text{colsp}(EA)$ which is the number of pivots.

Thus Theorem 5 allows us to define

$$\text{rank}(A) = \dim(\text{colsp}(A)) = \dim(\text{rowsp}(A)).$$

From this we obtain the following lovely result. It is often called the Nullity Theorem where nullity is $\dim(\text{nullsp}(A))$.

Theorem 6 Let $A$ be an $m \times n$ matrix. Then $\text{rank}(A) + \dim(\text{nullsp}(A)) = n$.

Proof: $\dim(\text{nullsp}(A))$ is the number of free variables. We have the number of pivot variables and the number of free variables is $n$. 

$\blacksquare$