We have indicated that determining if a set of n functions $\{f_1, f_2, \ldots, f_n\}$ is linearly independent is as easy as finding n values $x_1, x_2, \ldots, x_n$ in the domain and forming the matrix $A = (a_{ij})$ where $a_{ij} = f_j(x_i)$. If $\det(A) \neq 0$, then the n functions are linearly independent.

The idea of the wronskian is another way to check if a set of functions are linearly independent using derivatives. The applications are typically in differential equations for which derivatives are often easy to come by. Imagine we have n functions $\{f_1, f_2, \ldots, f_n\}$ satisfying

$$a_1 f_1 + a_2 f_2 + \cdots + a_n f_n = \mathbf{0}$$

Then, assuming the functions have the appropriate derivatives, we can differentiate repeatedly to have

$$\begin{array}{c} a_1 f_1 + a_2 f_2 + \cdots + a_n f_n = \mathbf{0} \\ a_1 f_1' + a_2 f_2' + \cdots + a_n f_n' = \mathbf{0} \\ a_1 f_1'' + a_2 f_2'' + \cdots + a_n f_n'' = \mathbf{0} \\ \vdots \\ a_1 f_1^{(n-1)} + a_2 f_2^{(n-1)} + \cdots + a_n f_n^{(n-1)} = \mathbf{0} \end{array}$$

where $f^{(i)}$ refers to the ith derivative of f. Now form the matrix $A(x) = (a_{ij})$ where the entries are functions $a_{ij} = f_j^{(i-1)}(x)$. The wronskian $W(x) = \det(A(x))$, which will be a function of x. Now if $W(x) \neq 0$ for some $x = c$, then the n functions $\{f_1, f_2, \ldots, f_n\}$ are seen to be linearly independent since if $a_1 f_1 + a_2 f_2 + \cdots + a_n f_n = \mathbf{0}$ then $A(c)\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = (a_1, a_2, \ldots, a_n)^T$. But $\det(A(c)) = W(c) \neq 0$ and so we conclude $a_1 = a_2 = \ldots = a_n = 0$ (since $(A(c))^{-1}$ exists). This shows that the n functions $\{f_1, f_2, \ldots, f_n\}$ are linearly independent.

An attractive application is for the function $f(x) = \frac{1}{x-r}$ for which

$$f^{(i)}(x) = \frac{(-1)^i i!}{(x-r)^{i+1}}$$

Are the n functions $\{\frac{1}{x-r_1}, \frac{1}{x-r_2}, \ldots, \frac{1}{x-r_n}\}$ linearly independent for n distinct choices of r_i? We compute that

$$W(x) = \det \begin{pmatrix} \frac{1}{x-r_1} & \frac{1}{x-r_2} & \cdots & \frac{1}{x-r_n} \\ \frac{-1}{(x-r_1)^2} & \frac{-1}{(x-r_2)^2} & \cdots & \frac{-1}{(x-r_n)^2} \\ \frac{2}{(x-r_1)^3} & \frac{2}{(x-r_2)^3} & \cdots & \frac{2}{(x-r_n)^3} \\ & \vdots & & \\ \frac{(-1)^{n-1}(n-1)!}{(x-r_1)^n} & \frac{(-1)^{n-1}(n-1)!}{(x-r_2)^n} & \cdots & \frac{(-1)^{n-1}(n-1)!}{(x-r_n)^n} \end{pmatrix}$$

and so

$$W(x) = \left((-1)^{(n-1)(n-2)/2} \prod_{i=1}^{n} s_i \prod_{i=1}^{n-1} (i)! \right) \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ s_1 & s_2 & \cdots & s_n \\ s_1^2 & s_2^2 & \cdots & s_n^2 \\ & \vdots & & \\ s_1^{n-1} & s_2^{n-1} & \cdots & s_n^{n-1} \end{pmatrix}$$

where $s_i = \frac{1}{x-r_i}$. We have pulled out a factor s_i from the ith column and a factor $(-1)^{n-1}(i-1)!$ from the ith row. We know that the determininant is a VanderMonde determinant and is non-zero since the s_i's are all distinct. Thus the n functions are linearly independent.