Math 340 Bland's Rule Richard Anstee

This result is due to Bob Bland of Cornell University. Bland's Rule is the pivot rule that is the same as Anstee's Rule except that when choosing an entering variable we *choose the variable of smallest subscript among all those with a positive coefficient in the z row.*

Theorem If we use Bland's Rule, we will not cycle.

Proof: Assume we do have cycling using Bland's rule with $B_1 \to B_2 \to B_3 \to \cdots \to B_\ell \to B_1$. Let D_i be the dictionary associated with basis B_i. Let a variable x_j be called *fickle* if $x_j \in B_g$ and yet $x_j \notin B_h$ for some pair $g, h \in \{1, 2, \ldots, \ell\}$. We can describe the fickle variables as precisely those either entering or leaving during the cycle. We already know in a cycle that each pivot will be a degenerate pivot (you should know why?) and hence, if x_j is fickle, $x_j = 0$ in each basic feasible solution during the cycle.

Let x_t be the fickle variable of largest subscript. Assume $x_t \in B_i$ and yet $x_t \notin B_{i+1}$. Assume x_s is the entering variable as we pass from B_i to B_{i+1}. Thus $\boxed{s < t}$. We write the dictionary D_i for B_i as follows

$$D_i \qquad \begin{cases} x_k &= b_k - \sum_{x_j \notin B_i} a_{kj} x_j \\ \\ z &= v + \sum_{x_j \notin B_i} c_j x_j \end{cases}$$

We have chosen useful terminology. We are not asserting that a_{kj} nor c_j arises from the original data. Since we chose x_s to enter we have $\boxed{c_s > 0}$. Now x_t leaving means that $\boxed{a_{ts} > 0}$ and $\boxed{b_t = 0}$. Now assume x_t reenters basis for the first time in the pivot from D_f.

$$D_f \qquad \begin{cases} \cdots &= \cdots \\ \\ z &= v + \sum_{x_j \notin B_f} c_j^* x_j \end{cases}$$

Again we have chosen useful terminology and are not asserting that somehow c_j^* is optimal, which wouldn't make much sense anyway. We deduce that $\boxed{c_t^* > 0}$. The equation for z in D_f is obtained from the equation from D_i by adding suitable multiples of equations of D_i to the z row. A solution of D_i which is **not necessarily feasible** is

$$\begin{aligned} x_s &= q \\ x_j &= 0 \text{ for } x_j \notin B_i \text{ and } j \neq s \\ x_k &= b_k - a_{ks} q \text{ for } x_k \in B_i \\ z &= v + c_s q \end{aligned}$$

As we showed early on, the set of solutions is preserved as we move from dictionary to dictionary (we are multiplying the set of equations on the left by an invertible matrix) and so these solution (one for each q) are also solutions to D_f. Thus

$$z = v + c_s q = v + c_s^* q + \sum_{x_k \in B_i} c_k^* (b_k - a_{ks} q)$$

where we set $c_k^* = 0$ for $x_k \in B_f$. We now have

$$\left(c_s - c_s^* + \sum_{x_k \in B_i} c_k^* a_{ks} \right) q = \sum_{x_k \in B_i} c_k^* b_k.$$

This above equation is true for all q and so we deduce

$$c_s - c_s^* + \sum_{x_k \in B_i} c_k^* a_{ks} = 0$$

Because x_s is not entering in D_f, we have $\boxed{c_s^* \leq 0}$ (if $c_s^* > 0$, then x_s would be chosen over x_t as the entering variable by Bland's Rule). Hence

$$\sum_{x_k \in B_i} c_k^* a_{ks} < 0$$

and so for some r with $x_r \in B_i$, we will have

$$c_r^* a_{rs} < 0.$$

Thus $c_r^* \neq 0$ and so $x_r \notin B_f$ and hence x_r is fickle. Now we have already found $c_t^* a_{ts} > 0$ and so $r \neq t$ and so by our choice of x_t we have $\boxed{r < t}$. Also x_r is not the entering variable in D_f and so $\boxed{c_r^* \leq 0}$. Thus $\boxed{a_{rs} > 0}$. (We also obtain $c_r^* < 0$). Now because x_r is fickle, we deduce that $\boxed{b_r = 0}$ in D_i. But now when we pivot in D_i we would preferentially choose x_r over x_t as the leaving variable, a contradiction. Thus cycling does not occur while using Bland's Rule.