

Proof of Complementary Slackness

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We will need Strong Duality to assert that if we have optimal solutions \mathbf{x}^* to the primal and \mathbf{y}^* to the dual then $\mathbf{c} \cdot \mathbf{x}^* = \mathbf{b} \cdot \mathbf{y}^*$.

Theorem (Complementary Slackness) Let \mathbf{x} be a feasible solution to the primal and \mathbf{y} be a feasible solution to the dual where

$$\begin{array}{ll} \max & \mathbf{c} \cdot \mathbf{x} \\ \text{primal} & \begin{array}{l} A\mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{array} \end{array} \quad \begin{array}{ll} \min & \mathbf{b} \cdot \mathbf{y} \\ \text{dual} & \begin{array}{l} A^T \mathbf{y} \geq \mathbf{c} \\ \mathbf{y} \geq \mathbf{0} \end{array} \end{array}.$$

Then \mathbf{x} is optimal to the primal and \mathbf{y} is optimal to the dual if and only if the conditions of Complementary Slackness hold:

$$(b_i - \sum_{j=1}^n a_{ij}x_j)y_i = x_{n+i}y_i = 0 \text{ for } i = 1, 2, \dots, m$$

and

$$\left(\sum_{i=1}^m a_{ji}y_i - c_j\right)x_j = y_{m+j}x_j = 0 \text{ for } j = 1, 2, \dots, n.$$

Proof: Given that \mathbf{x} and \mathbf{y} are feasible implies $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, $A^T \mathbf{y} \geq \mathbf{c}$, and $\mathbf{y} \geq \mathbf{0}$. We now use the ideas in our proof of Weak Duality. Now $\mathbf{x} \geq \mathbf{0}$, $A^T \mathbf{y} \geq \mathbf{c}$ implies $\mathbf{x}^T A^T \mathbf{y} \geq \mathbf{x}^T \mathbf{c}$ where $\mathbf{x}^T \mathbf{c} = \mathbf{c} \cdot \mathbf{x}$. Also $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{y} \geq \mathbf{0}$ implies $\mathbf{y}^T A\mathbf{x} \leq \mathbf{y}^T \mathbf{b}$ where $\mathbf{y}^T \mathbf{b} = \mathbf{b} \cdot \mathbf{y}$. We have

$$\mathbf{c} \cdot \mathbf{x} = \mathbf{x}^T \mathbf{c} \leq \mathbf{x}^T A^T \mathbf{y} = \mathbf{y}^T A\mathbf{x} \leq \mathbf{y}^T \mathbf{b} = \mathbf{b} \cdot \mathbf{y}$$

using the fact that $\mathbf{x}^T A^T \mathbf{y}$ is a 1×1 matrix and so $\mathbf{x}^T A^T \mathbf{y} = (\mathbf{x}^T A^T \mathbf{y})^T = \mathbf{y}^T A\mathbf{x}$. Now by Strong Duality, \mathbf{x} and \mathbf{y} are both optimal to their respective Linear Programs if and only if $\mathbf{c} \cdot \mathbf{x} = \mathbf{b} \cdot \mathbf{y}$ and so if and only if $\mathbf{x}^T \mathbf{c} = \mathbf{x}^T A^T \mathbf{y}$ and $\mathbf{y}^T A\mathbf{x} = \mathbf{y}^T \mathbf{b}$ and hence if and only if $\mathbf{x} \cdot (A^T \mathbf{y} - \mathbf{c}) = 0$ and $\mathbf{y} \cdot (\mathbf{b} - A\mathbf{x}) = 0$. We note that $0 = \mathbf{x} \cdot (A^T \mathbf{y} - \mathbf{c}) = \sum_{j=1}^n \left(x_j (\sum_{i=1}^m a_{ji}y_i - c_j) \right)$ and since $x_j \geq 0$ and $\sum_{i=1}^m a_{ji}y_i - c_j \geq 0$ for each $j = 1, 2, \dots, n$ we deduce that $x_j (\sum_{i=1}^m a_{ji}y_i - c_j) \geq 0$ for each $j = 1, 2, \dots, n$. But then each of the n terms in the sum $\sum_{j=1}^n \left(x_j (\sum_{i=1}^m a_{ji}y_i - c_j) \right)$ are positive and yet the sum is 0 and so we deduce that each term is 0, namely $x_j (\sum_{i=1}^m a_{ji}y_i - c_j) = 0$ for each $j = 1, 2, \dots, n$ which is the second condition for Complementary Slackness. Similarly we deduce that $\mathbf{y} \cdot (\mathbf{b} - A\mathbf{x}) = 0$ if and only if

$$y_i (b_i - \sum_{j=1}^n a_{ij}x_j) = y_i x_{n+i} = 0 \text{ for } i = 1, 2, \dots, m.$$

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