Math 340 **Some Brief Game Theory Notes**
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This material can be found in the text in greater detail in Chapter 15 which is certainly worth reading. Our battleship problem follows from an exercise in the text. The game of Morra and its variant where Claude (the column player) reveals his guess before Trucula are well explained in the text.

We discuss *two person zero sum games*. This means there are two players *player 1* and *player 2* and the payoff to player 2 in the negative of the payoff to player 1. Thus the net payoff to both players in total is zero. There are games with a positive payoff (e.g. stock market) and a negative payoff (e.g. lotteries). One assumes that Player 1 has m strategies amd that Player 2 has n strategies and that there is an $m \times n$ payoff matrix $A = (a_{ij})$ where

$$a_{ij} = \text{payoff to player 1 if player 1 uses strategy } i \text{ and player 2 uses strategy} j.$$

As a result of this matrix interpretation, we often refer to player 1 as the *row player* and player 2 as the *column player*. For example, if player 1 plays strategy 3 then the game reverts to just row 3 of the payoff matrix A as far as player 2 is concerned. A *mixed strategy* for player 1 corresponds to a vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)^T$ where $\mathbf{x}$ satisfies $\mathbf{x} \geq \mathbf{0}$ and $x_1 + x_2 + \cdots + x_m = 1$. Thus x_i is the probability of player 1 playing strategy i.

If player 1 (the row player) plays the (mixed) strategy x, then as far as player 2 (the column player) is concerned the game has the *expected* payoff matrix $\mathbf{x}^T A$.

Note that if player 1 (the row player) plays the pure strategy i, then as far as player 2 (the column player) is concerned the game has been reduced to the ith row of A. Then the optimal strategy for player 2 is to choose the minimum entry in $\mathbf{x}^T A$ (the minimum in the payoff matrix maximizes the payoff to player 2; this confusion that the payoff to player 2 is the negative of the expected payoff to player 1 will reoccur many times in this theory) which, given that player 1 plays the mixed strategy $\mathbf{x}$, is then an lower bound on the expected value of the payoff to Player 1. Note that we are assuming player 2 can't cheat and somehow predict player 1's choices. We are assuming player 1 plays the first (pure) strategy with probability x_1. If for example $x_1 = .5$, we are not imagining that (row) player 1 plays (pure) strategy 1 every second game but chooses strategy 1 with probability 1/2.

We will often simply refer to a *mixed strategy* $\mathbf{x}$ as a *strategy* even though it corresponds to choosing the ith strategy with probability x_i. We refer to a *pure strategy* (one of the strategies given as a row index of A) as a mixed strategy with one entry equal to 1 and necessarily all other entries are 0.

In a similar way, if y is the mixed strategy for the column player, then as far as the row player is concerned, the game has the payoff matrix $A\mathbf{y}$.

You may envision other possible ways to approach player strategies such as adapting your strategy as play unfolds and you witness more of your opponents strategies. This is easily attempted with small children when playing Rock-Scissors-Paper. When first learning the game they may follow a predictable set of moves based on your moves. If you beat them scissors over paper, the next round you can predict they may try scissors and you can beat them by playing rock. This kind of play is not the subject of the game theory here.

Let z be the minimum entry of $\mathbf{x}^T A$:

$$z = \min\{a_{11}x_1 + a_{21}x_2 + \cdots + a_{m1}x_m, \quad a_{12}x_1 + a_{22}x_2 + \cdots + a_{m2}x_m,$$

$$\ldots, a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{mn}x_m\}$$

We wish to maximize z over all choices of strategies $\mathbf{x}$. We cannot directly create a linear constraint yielding the minimum but instead impose m constraints $z \le i$th entry of $\mathbf{x}^T A$ which makes z less than or equal to the minimum but then since we maximize z, we get z equal to the minimum. This particular idea works in some other practical problems.

$$LP1 \quad \begin{array}{lrrll} \max & & z & & \\ & -A^T\mathbf{x} & +\mathbf{1}z & \le 0 & \quad \mathbf{x} \ge \mathbf{0}, z \text{ free .} \\ & \mathbf{1}^T\mathbf{x} & & = 1 & \end{array}$$

More explicitly, the constraints are:

$$LP1 \quad \begin{array}{llll} \max & & z & \\ & -a_{11}x_1 - a_{21}x_2 - \cdots - a_{m1}x_m & +z & \le 0 \\ & -a_{12}x_1 - a_{22}x_2 - \cdots - a_{m2}x_m & +z & \le 0 \\ & \vdots & & \\ & -a_{1n}x_1 - a_{2n}x_2 - \cdots - a_{mn}x_m & +z & \le 0 \\ & x_1 + x_2 + \cdots + x_m & & = 1 \end{array} \quad x_1, x_2, \ldots, x_m \ge 0, z \text{ free .}$$

It is potentially confusing that the first game of Morra discussed in the text is a symmetric game and so we have $-A^T = A$ which can cause confusion to students. Our first example below does not have this property so as to avoid this confusion.

This is what I call a *conservative approach* for player 1. The optimal value is the guaranteed (expected) payoff to player 1 regardless of player 2's strategy (pure or mixed). For a payoff matrix

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 6 & 4 & 2 \end{bmatrix},$$

the LP becomes

$$\begin{array}{lrrrl} \max & & & z & \\ & -2x_1 & -6x_2 & +z & \le 0 \\ & -3x_1 & -4x_2 & +z & \le 0 \\ & -4x_1 & -2x_2 & +z & \le 0 \\ & x_1 & +x_2 & & = 1 \end{array} \quad \mathbf{x} \ge \mathbf{0}, z \text{ free .}$$

The dual is

$$LP2 \quad \begin{array}{lrrll} \min & & w & & \\ & -A\mathbf{y} & +\mathbf{1}w & \ge 0 & \quad \mathbf{y} \ge \mathbf{0}, w \text{ free .} \\ & \mathbf{1}^T\mathbf{y} & & = 1 & \end{array}$$

More explicitly, the dual is

$$LP2 \quad \begin{array}{llll} \min & & w & \\ & -a_{11}y_1 - a_{12}y_2 + \cdots - a_{1n}y_n & +w & \ge 0 \\ & -a_{21}y_1 - a_{22}y_2 + \cdots - a_{2n}y_n & +w & \ge 0 \\ & \vdots & & \\ & -a_{m1}y_1 - a_{m2}y_2 + \cdots - a_{mn}y_n & +w & \ge 0 \\ & y_1 + y_2 + \cdots + y_n & & = 1 \end{array} \quad \mathbf{y} \ge \mathbf{0}, w \text{ free .}$$

Using our standard transformations, this is equivalent to

$$LP3 \quad \begin{array}{lrrll} \max & & -w & & \\ & A\mathbf{y} & -\mathbf{1}w & \le 0 & \quad \mathbf{y} \ge \mathbf{0}, w \text{ free .} \\ & \mathbf{1}^T\mathbf{y} & & = 1 & \end{array}$$

which we can see is the LP for determining player 2 optimal strategy after replacing $-w$ by a new variable, say v, and noting that the payoff matrix for player 2 (when viewed as a player 1 or row player) is $-A^T$ (rows now to correspond to player 2 strategies and columns to player 1 strategies and the entries are the payoff to player 2) and noting that $-(-A^T)^T = A$. Thus the value of LP2 is the most player 2 can expect to lose which is again a conservative estimate. Now Strong Duality shows that these conservative approaches are best possible.

Theorem (Von Neumann 1929). Given an $m \times n$ matrix A corresponding to the payoff matrix to player 1 for a two person zero sum game, there is a strategy $\mathbf{x}^*$ for player 1 and strategy $\mathbf{y}^*$ for player 2 with

$$\min_{\mathbf{y}:\mathbf{y} \text{ is strategy for player } 2} (\mathbf{x}^*)^T A\mathbf{y} = \max_{\mathbf{x}:\mathbf{x} \text{ is strategy for player } 1} \mathbf{x}^T A\mathbf{y}^* = v(A)$$

where now $v(A)$ denotes the *value* of the game.

This is seen to be a consequence of Strong Duality once you can assert one of the possible hypotheses to the Strong Duality Theorem, namely either x^* exists or $\mathbf{y}^*$ exists or both LP1 and LP2 have feasible solutions. The last of the three condition is easy to see. We can get feasible solutions quite easily to both LP's. For LP1 we can take $x_1 = 1, x_2 = x_3 = \cdots = x_m = 0$ and then appropriately choose a value for z, say take z to be the minimum of $\mathbf{x}^T A$.

Returning to the particular example above,

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 6 & 4 & 2 \end{bmatrix},$$

we see that by taking the mixed (1/2,1/2) we have reduced the game as far as player 2 is concerned to [4 3.5 3] for which player 2 can ensure a loss of no more than 3. We can do better for player 1 with the strategy (2/3,1/3) which reduces the game to [10/3 10/3 10/3] for which player 2 can ensure a loss of no more than 10/3. Is this best possible? By using Complementary Slackness, or other techniques, we can find a strategy for player 2. The dual problem is

$$\begin{array}{lrrrrl} \min & & & w & & \\ & -2y_1 & -3y_2 & -4y_3 & +w & \geq 0 \\ & -6y_1 & -4y_2 & -2y_3 & +w & \geq 0 \\ & y_1 & +y_2 & +y_3 & & = 1 \end{array} \qquad \mathbf{y} \geq \mathbf{0}, w \text{ free .}$$

Using $x_1 = 2/3$, we deduce that $2y_1 + 3y_2 + 4y_3 = 10/3$. Using $x_2 = 1/3$, we deduce that $6y_1 + 4y_2 + 2y_3 = 10/3$. Also we know that $y_1 + y_2 + y_3 = 1$. The system of three equations in three unknowns solves to

$$\begin{array}{rrrcr} y_1 & +y_2 & +y_3 & = & 1 \\ & y_2 & +2y_3 & = & 4/3 \end{array}$$

which yields solutions

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 4/3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \text{ for suitable choices of } t \text{ maintaining feasibility.}$$

We get such a vector parametric form because one of the equations is a linear combination of the other two. Now $\mathbf{y} \geq \mathbf{0}$ and so we deduce that we have feasible and hence optimal solutions for $1/3 \leq t \leq 2/3$. Thus we have an infinite number of dual optimal solutions (as we had in Morra; it is not that unusual) yielding a payoff to player 1 of at most 10/3. Thus $v(A) = 10/3$.

A Game is said to be *fair* if $v(A) = 0$.

Theorem If the payoff matrix satisfies $-A^T = A$, then the game is fair, namely $v(A) = 0$.

A proof of this follows by showing that $v(A) = -v(A)$ by showing that player 1 (the row player) and player 2 (the column player) are solving the same LP's.

Of course other games not satisfying $-A^T = A$ can be fair. Moreover if A has $v(A) = k$, then we can get a fair game by requiring player 1 to pay player 2 an amount of k at the start of the game for the privilege of playing the 'unfair' game and so the total payoff matrix becomes $A - kJ$ where J is the appropriately sized matrix of 1's and $v(A - kJ) = 0$.