Math 340 **Two Person Zero Sum Games correspond to LP Theory**
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This material is a problem from the text. Consider our standard primal dual pair:

$$\text{primal:}\quad \begin{array}{rl} \max & \mathbf{c}\cdot\mathbf{x} \\ & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0} \end{array} \qquad \text{dual:}\quad \begin{array}{rl} \min & \mathbf{b}\cdot\mathbf{y} \\ & A^T\mathbf{y} \ge \mathbf{c} \\ & \mathbf{y} \ge \mathbf{0} \end{array}$$

and consider the payoff matrix B as follows

$$B = \begin{bmatrix} 0 & -A^T & \mathbf{c} \\ A & 0 & -\mathbf{b} \\ -\mathbf{c}^T & \mathbf{b}^T & 0 \end{bmatrix}.$$

If A is of size $m\times n$, then B is of size $(n+m+1)\times(n+m+1)$. We note that $B^T = -B$ and so B is skew symmetric and so $v(B) = 0$.

Theorem 0.1 *The primal and dual have optimal solutions if and only if the game given by payoff matrix B has an optimal mixed strategy $\mathbf{u}^*$ with the last strategy being non zero, namely $u^*_{n+m+1} > 0$.*

Proof: Assume A is size $m \times n$. With $v(B) = 0$, there is an optimal strategy $\mathbf{u}^*$ for the row player which is also an optimal strategy for the column player. We have

minimum entry of $(\mathbf{u}^*)^T B = 0 =$ maximum entry of $B\mathbf{u}^*$.

Now assume the primal dual pair have optimal solutions $\mathbf{x}^* = (x_1^*, x_2,^*, \dots, x_n^*)^T$ and $\mathbf{y}^* = (y_1^*, y_2^*, \dots, y_m^*$. Let

$$t = \frac{1}{\sum_j x_j^* + \sum_i y_i^* + 1}$$

Set $\bar{\mathbf{x}} = t\mathbf{x}^*$ and $\bar{\mathbf{y}} = t\mathbf{y}^*$. Let

$$\mathbf{u}^* = \begin{bmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{y}} \\ t \end{bmatrix}$$

I claim $\mathbf{u}^*$ is an optimal strategy for either player. We note that $\mathbf{x}^* \ge \mathbf{0}$ and $\mathbf{y}^* \ge \mathbf{0}$ and definition of t yields $\mathbf{u}^* \ge \mathbf{0}$. Also using the definition of t we have $\sum_k u_k^* = 1$. We compute

$$(\mathbf{u}^*)^T B = [(\bar{\mathbf{y}})^T A - t\mathbf{c}^T, -(\bar{\mathbf{x}})^T A^T + t\mathbf{b}^T, \bar{\mathbf{x}}^T\mathbf{c} - \bar{\mathbf{y}}^T\mathbf{b}]$$

we have

$$(\bar{\mathbf{x}})^T\mathbf{c} - (\bar{\mathbf{y}})^T\mathbf{b} = t((\mathbf{x}^*)^T\mathbf{c} - (\mathbf{y}^*)^T\mathbf{b}) = 0,$$

using Strong Duality. Now $A^T\mathbf{y}^* \ge \mathbf{c}$ so that $(\mathbf{y}^*)^T A \ge \mathbf{c}^T$ and so $(\mathbf{y}^*)^T A - \mathbf{c}^T \ge \mathbf{0}^T$. Thus

$$(\bar{\mathbf{y}})^T A - t\mathbf{c}^T = t(\mathbf{y}^*)^T A - t\mathbf{c}^T = t((\mathbf{y}^*)^T A - \mathbf{c}^T) \ge \mathbf{0}^T.$$

Similarly $A\mathbf{x} \le \mathbf{b}$ so that $(\mathbf{x}^*)^T A^T \le \mathbf{b}^T$ and so $-\mathbf{x}^*)^T A^T + \mathbf{b}^T \ge \mathbf{0}^T$. Thus

$$-(\bar{\mathbf{x}})^T A^T + t\mathbf{b}^T = -t(\mathbf{x}^*)^T A^T + t\mathbf{b}^T = t((-\mathbf{x}^*)^T A^T + \mathbf{b}^T) \ge \mathbf{0}^T,$$

Now we also have $\bar{\mathbf{x}}^T\mathbf{c} - \bar{\mathbf{y}}^T\mathbf{b} = t((\mathbf{x}^*)^T\mathbf{c} - (\mathbf{y}^*)^T\mathbf{b}) = 0$ by strong duality since $\mathbf{x}^*$ and $\mathbf{y}^*$ are optimal to their respective LP's. This proves that $(\mathbf{u}^*)^T B \ge \mathbf{0}$ and this is enough to make $\mathbf{u}^*$ optimal in view of $v(B) = 0$. Moreover $\mathbf{u}^*$ has its last entry $t > 0$. This completes the 'only if' half of the if and only if proof.

Now assume we have an optimal solution $\mathbf{u}^*$ to the game given by B where

$$\mathbf{u}^* = \begin{bmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{y}} \\ t \end{bmatrix}$$

We have $\bar{\mathbf{x}} \ge \mathbf{0}$ and $\bar{\mathbf{y}} \ge \mathbf{0}$ and $u^*_{m+n+1} = t > 0$. We claim we obtain optimal solutions to the primal dual pair by setting

$$\mathbf{x}^* = t\bar{\mathbf{x}}, \qquad \mathbf{y}^* = t\bar{\mathbf{y}}.$$

Given $v(B) = 0$, we have 0 is the minimum entry of $(\mathbf{u}^*)^T B$. Thus
$(\bar{\mathbf{y}})^T A - t\mathbf{c}^t \ge \mathbf{0}$
$(-\bar{\mathbf{x}})^T A^T + t\mathbf{b}^T \ge \mathbf{0}$
$(\bar{\mathbf{x}})^T\mathbf{c} - (\bar{\mathbf{y}})^T\mathbf{b} \ge 0$
Rewriting in terms of $\mathbf{x}^*$, $\mathbf{y}^*$ (and using $t > 0$), we obtain
$(\mathbf{y}^*)A - \mathbf{c}^T \ge \mathbf{0}$ which is $A^T\mathbf{y}^* \ge c$,
$(-x^*)^T A^T + \mathbf{b} \ge \mathbf{0}$ which is $A\mathbf{x}^* \le \mathbf{b}$.
$(\mathbf{x}^*)^T\mathbf{c} - (\mathbf{y}^*)^T\mathbf{b} \ge 0$ which is $\mathbf{x}^* \cdot \mathbf{c} \ge \mathbf{y}^* \cdot \mathbf{b}$.
We have $\bar{\mathbf{x}} \ge \mathbf{0}$ and $\bar{\mathbf{y}} \ge \mathbf{0}$ and so $\mathbf{x}^* \ge \mathbf{0}$ and $\mathbf{y}^* \ge \mathbf{0}$. Thus $\mathbf{x}^*$ is feasible to the primal and $\mathbf{y}^*$ is feasible to the dual. Weak duality gives us that $\mathbf{x}^* \cdot \mathbf{c} \le \mathbf{y}^* \cdot \mathbf{b}$ with equality if and only if $\mathbf{x}^*$ is optimal to the primal and $\mathbf{y}^*$ is optimal to the dual. But we already have that $\mathbf{x}^* \cdot \mathbf{c} \ge \mathbf{y}^* \cdot \mathbf{b}$ and so $\mathbf{x}^* \cdot \mathbf{c} = \mathbf{y}^* \cdot \mathbf{b}$. We now conclude that $\mathbf{x}^*$ is optimal to the primal and $\mathbf{y}^*$ is optimal to the dual completing the 'if' portion of the proof. ■